Raza Tahir-Kheli

Ordinary Differential Equations

Mathematical Tools for Physicists

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Dedicated to my friends
Sir Roger J. Elliott, Kt., F.R.S.
(in Memorium)
Sir Anthony J.Leggett, KBE, F.R.S., Nobel Laureate
Alan J. Heeger, Nobel Laureate
J. Robert Schrieffer, Nobel Laureate
my wife
Ambassador Shirin Raziuddin Tahir-Kheli
and our grandchildren
Taisiya, Alexander, Cyrus, Gladia

Preface

This book is intended as a reader-friendly source for self-study and as an accessible textbook that contains many solved problems.

My experience teaching at Temple University the mathematical physics course that includes ordinary differential equations has guided the writing of this textbook. The mathematical physics course is offered to undergraduates in their pre- or final year of study in physics, engineering, chemistry, earth and environmental sciences, or mathematical biology. It is also taken by beginning graduate students working toward a master's degree.

Years of teaching have helped me understand what works for students and what does not. In particular, I have learned that the more attention a student pays to taking notes, the less he/she understands of the subject matter of the lecture being delivered. Further, I have noticed that when, a week in advance of the delivery of the lecture, a student is provided details of the algebra to be used, solutions to the problems to be discussed, and some brief information about the ideas that are central to the lecture to be given the following week, it obviates much of the need for note taking during the delivery of the lecture. Another important experience that has guided the writing of this textbook is the pedagogical benefit that accrues from an occasional, quick, recapitulation of the relevant results that have already been presented in an earlier lecture. All this results in better comprehension of the subject matter.

Both for the purposes of elucidation of the concepts introduced in the text and for providing practical problem solving support, solutions to a large number of examples have been included. Many of the solutions provided contain much greater detail than would be necessary for presentation in a lecture itself or needed by teachers or more advanced practitioners of the subject. These solutions are there in the given form to offer encouragement and support to the student: both for self-study and to allow for fuller understanding of the subject matter. Therefore, it is as important for the reader to assimilate the subject matter of the book as it is to independently work through the solved problems before reading through the solutions provided. Another notable feature of the book is that equations are

numbered in seriatim. Included in the numbering process is the chapter number. As a result, access to an equation being referred to is both easy and fast.

In Chap. 1, the differential operator D, is introduced and the rules it follows are articulated.

A cursory look at the vocabulary is taken in Chap. 2 titled 'Some Definitions.' Noted there is the terminology and some of the definitions that are used in this manuscript. A function involving a dependent variable, say u, only a single independent variable, say x, and at least one derivative of u with respect to x is an ordinary differential equation (ODE). Explicit (ODE) and implicit (ODE) are defined. Generally, explicit (ODEs) are easier to treat than implicit (ODEs). Therefore, mostly the explicit (ODEs) are treated in this book. A linear ordinary differential equation is of first degree in the variable, $u(x)$, as well as its derivatives. Homogeneous linear ordinary differential equation and inhomogeneous linear ordinary differential equation are defined. When an ODE cannot be expressed in linear form, it is said to be nonlinear (ODE). Just as implicit (ODE) is harder to solve than explicit (ODE), solving nonlinear (ODE) requires more effort than solving linear (ODE). Furthermore, simple treatment of nonlinear (ODE) cannot be guaranteed to succeed.

Linear ordinary differential equations with known constant coefficients are treated in Chap. 3. Procedure for solving homogeneous linear ordinary differential equations with known constant coefficients, the method of undetermined coefficients, and calculation of the particular integral, I_{pi}, for inhomogeneous linear ordinary differential equations are all described in detail. The concepts of linear independence, linear dependence, and the use of Wronskians are elucidated. And simultaneous linear ordinary differential equations with constant coefficients are studied in detail.

In Chap. 4, the analysis is extended to linear ordinary differential equations that have variable coefficients. Depending upon the nature of the variable coefficient, the linear ordinary differential equation with variable coefficients can be much harder to solve than linear ordinary differential equations with constant coefficients. For simplicity, therefore, only the first-order and first-degree equations of type Bernoulli equation are treated. Included also is a discussion of equations that can be transformed into Bernoulli equation.

Chapter 5 deals with Green's function and Laplace transforms.

In Chap. 6, equations Beyond Bernoulli—to be called 'Special Type' differential equations—are studied. Included there are the Clairaut equations [compare (6.2)–(6.13)], Lagrange equation [compare (6.19)–(6.31)], the separable equations [compare (6.32)–(6.35)], and the $\frac{dy}{dx} = \Phi\left(\frac{y}{x}\right)$ equations [compare (6.36)–(6.73)]. In addition, there are the so-called exact [compare (6.74)–(6.91)] and inexact equations [compare (6.92)–(6.241)], Riccati equations [compare (6.242)–(6.268)], Euler equations [compare (6.269)–(6.315)], and the factorable equations [compare (6.316)–(6.344)]. Singular solution of Clairaut equation is discussed and calculated both by an informal procedure and a formal procedure.

Studied in Chap. 7 are equations where 'Special Situations' obtain. For instance, a given differential equation may be integrable. Similarly, there are equations that have both the independent and the dependent variables missing in any explicit form. And then, there are those that explicitly contain only the independent variable, or only the dependent variable. All these equations provide possible routes to successful treatment. An interesting special situation, called 'Order Reduction,' comes into play if one of the n non-trivial solutions of an nth order homogeneous linear ordinary differential equation is already known. Then, the given equation can be reduced to one of the $(n-1)$th order. Homogeneous linear ordinary second-order differential equations have been treated before. Studied in this chapter are inhomogeneous linear ordinary differential equations.

Chapter 8 deals with oscillatory motion that is central to the description of acoustics and the effects of inter-particle interaction in many physical systems. In its most accessible form, oscillatory motion is simple harmonic (s-h). (s-h) motion has a long and distinguished history of use in modeling physical phenomena. An harmonic motion—which somewhat more realistically represents the observed behavior of physical systems—is described next. Detailed analysis of 'Transient State' motion is presented for a point mass for two different oscillatory systems. These are:

(1) The point mass, m, is tied to the right end of a long, massless coil spring placed horizontally in the x-direction on top of a long, level, table. The left end of the coil is fixed to the left end of the long table. The motion of the mass is slowed by frictional force that is proportional to its momentum $mv(t)$. In its completely relaxed state, the spring is in equilibrium and the mass is in its equilibrium position.
(2) Because the differential equations needed for analyzing damped oscillating pendula are prototypical of those used in studies of electromagnetism, acoustics, mechanics, chemical and biological sciences and engineering, we analyze next a pendulum consisting of a (point-sized) bob of mass m that is tied to the end of a massless stiff rod of length l. The rod hangs down, in the negative z-direction, from a hook that has been nailed to the ceiling. The pendulum is set to oscillate in two-dimensional motion in the $x-z$ plane. Air resistance is approximated as a frictional force proportional to the speed with which the bob is moving. The ensuing friction slows the oscillatory motion.

In Chap. 9, physics relating to, and the applicable mathematics for the use of, resistors, inductors, and capacitors are studied. The study includes Kirchhoff's two rules that state: 'The incoming current at any given point equals the outgoing current at that point' and 'The algebraic sum of changes in potential encountered by charges traveling, in whatever manner, through a closed-loop circuit is zero.' Considered next is Ohm's law: namely 'In a closed-loop circuit that contains a battery operating at V volt, and a resistor of strength R ohms, current flow is I amperes: $I = \frac{V}{R}$.' Problems relating to additions of finite numbers of resistors, placed in various configurations—some in series and some in parallel formats—are

worked out in detail. Also included are several, more involved, analyses relating to current flows in a variety of infinite networks.

Numerical solutions are analyzed in Chap. 10. Given a first-order linear differential equation [refer to (10.2)] and its solution, $y(x_0)$, at a point $x = x_0$, Runge–Kutta procedure is used to calculate $Y(x_0 + \Delta)$, an estimate of the exact result $y(x_0 + \Delta)$. Runge-Kutta estimate is the least accurate when it uses only one step for the entire move. And indeed, as shown in (10.11), the single-step process does yield grossly inaccurate results. The two-step process—see (10.12) and (10.13)—improves the results only slightly. But the four-step effort—see (10.14)–(10.17)—does much better. It reduces the error to about $\left(\frac{1}{50}\right)$th of that for the one-step process.

Estimates from a ten-step Runge-Kutta process are recorded in table (10.1). These estimates—being in error only by $\left(100 \times \frac{0.0025}{22.17}\right) = 0.0113\%$—are highly accurate.

Coupled first-order differential equations (10.18) are treated next.

Together these equations are equivalent to a single second-order differential equation. Tables (10.2)–(10.7) display numerical results, X_n and Y_n, for the one-step, two-step, and the five-step processes. Table (10.8) contains numerical results X_n gathered during a twenty-step Runge-Kutta process. At maximum extension, $\Delta = 2$, the Runge-Kutta estimate X_n is 26.371190. It differs from the exact result, 26.371404, by only a tiny amount, 0.000214. The percentage error involved is 0.000811.

Table (10.9) records numerical results Y_n collected during a twenty-step Runge-Kutta process. At maximum extension, $\Delta = 2$, the Runge-Kutta estimate Y_n is 11.0170347, It differs from the exact result, 11.0171222, by 0.0000875. The percentage error involved is 0.000794. It is similar to the corresponding error, 0.000811%, for X_n. The accuracy achieved by the twenty-step Runge-Kutta estimate is quite extraordinary. When very high accuracy is desired, the twenty-step Runge-Kutta estimate yields results that are worth the effort.

Chapter 11 deals with Frobenius' work. As stated earlier,

linear (ODEs) with variable coefficients are generally hard to treat. Fortunately, Frobenius' method may often be helpful in that regard. To that purpose, analytic functions, ordinary points, and regular and irregular singular points are described in this chapter. Frobenius assumes a modified Taylor series solution that is valid in the neighborhood of an ordinary point. The unknown constants there are determined through actual use of the Taylor series solution.

Frobenius solution around ordinary point is worked out for differential equations of type (a)—see (11.5)–(11.30)—and differential equations of type (b)—see (11.32)–(11.60).

Equations of type (c), (11.62), around regular singular points are treated next–see (11.63)–(11.75).

Indicial equation is defined—see (11.76)—and equations of category (1), whose two roots differ by non-integers, of category (2), whose two roots are equal, and of category (3), whose two roots differ by an integer, are all analyzed [See (11.78)–(11.183)].

The last part of Chap. 9 deals with Bessel's equations. Details of the relevant analyses are provided in (11.184)–(11.242).

Answers to assigned problems are given in Chap. 12.

Fourier transforms and *Dirac*[33.] delta function are treated in the appendix which forms Chap. 13.

Bibliography is presented last.

Unlike a novel, which is often read continuously—and the reading is completed within a couple of days—this book is likely to be read piecemeal—a chapter or two a week. At such slow rate of reading, it is often hard to recall the precise form of a relationship that appeared in the previous chapter. To help relieve this difficulty, when needed, the most helpful explanation of the issue at hand is repeated briefly and the most relevant expressions are referred to by their equation numbers. Throughout the book, for efficient reading, most equations are numbered in seriatim. When needed, they can be accessed quickly.

Most of the current knowledge of differential equations is much older than those of us who study it. The present book owes in motivation to a famous treatise by *Piaggio*[10.]—first published nearly a century ago by G. Bell and Sons, LTD., and last reprinted in (1940). Piaggio is a great book, but in some important places it misses, and sometime misprints, relevant detail. Numerous other *books*[11.–21.] of varying usefulness are also available. The current text stands apart from these books in that it is put together with a view to being accessible to all interested readers: for use both as a textbook and a book for self-study.

Answers to problems suggested for solution are appended in Chap. 12.

Finally, but for the support of my colleague Robert Intemann, this book could not have been written.

Philadelphia, USA Raza Tahir-Kheli
May 2018

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Chapter 1
Differential Operator



1.1 D

All differential equations make use of the differential operator D. A convenient notation for it is the following:

$$D^0 = 1 \ ; \ \ D^1 \ = \ D = \frac{\mathrm{d}}{\mathrm{d}x} \ ;$$
$$D^\nu = \frac{\mathrm{d}^\nu}{\mathrm{d}x^\nu} \ , \ \ \nu = 2, 3, 4, \ldots \tag{1.1}$$

The operator D follows some, but not all, laws of algebra.

1.1.1 Laws of Addition

Assume ν, ν_1, and ν_2 are positive integers. Using the rules of calculus, we can check that D obeys the commutative law of addition,

$$\frac{\mathrm{d}^{\nu_1} f(x)}{\mathrm{d}\, x^{\nu_1}} + \frac{\mathrm{d}^{\nu_2} f(x)}{\mathrm{d}\, x^{\nu_2}} = \frac{\mathrm{d}^{\nu_2} f(x)}{\mathrm{d}\, x^{\nu_2}} + \frac{\mathrm{d}^{\nu_1} f(x)}{\mathrm{d}\, x^{\nu_1}}$$
$$= (D^{\nu_1} + D^{\nu_2})\, f(x) = (D^{\nu_2} + D^{\nu_1})\, f(x) \ , \tag{1.2}$$

as well as the distributive law of addition–multiplication.

$$\frac{\mathrm{d}^\nu}{\mathrm{d}\, x^\nu} \left(\frac{\mathrm{d}^{\nu_1}}{\mathrm{d}\, x^{\nu_1}} + \frac{\mathrm{d}^{\nu_2}}{\mathrm{d}\, x^{\nu_2}} \right) f(x) = \frac{\mathrm{d}^\nu}{\mathrm{d}\, x^\nu} \frac{\mathrm{d}^{\nu_1}}{\mathrm{d}\, x^{\nu_1}} f(x) + \frac{\mathrm{d}^\nu}{\mathrm{d}\, x^\nu} \frac{\mathrm{d}^{\nu_2}}{\mathrm{d}\, x^{\nu_2}} f(x)$$
$$= D^\nu \left(D^{\nu_1} + D^{\nu_2}\right) f(x) = D^\nu D^{\nu_1} f(x) + D^\nu D^{\nu_2} f(x) \ . \tag{1.3}$$

Also the associative law of addition is obeyed.

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$$\left[\left(\frac{\mathrm{d}^{\nu}}{\mathrm{d}\,x^{\nu}}+\frac{\mathrm{d}^{\nu_1}}{\mathrm{d}\,x^{\nu_1}}\right)+\frac{\mathrm{d}^{\nu_2}}{\mathrm{d}\,x^{\nu_2}}\right] f(x)=\left[\frac{\mathrm{d}^{\nu}}{\mathrm{d}\,x^{\nu}}+\left(\frac{\mathrm{d}^{\nu_1}}{\mathrm{d}\,x^{\nu_1}}+\frac{\mathrm{d}^{\nu_2}}{\mathrm{d}\,x^{\nu_2}}\right)\right] f(x)$$
$$=\left[(D^{\nu}+D^{\nu_1})+D^{\nu_2}\right] f(x)=\left[D^{\nu}+(D^{\nu_1}+D^{\nu_2})\right] f(x). \tag{1.4}$$

1.1.2 Laws of Multiplication

Similarly, both the associative law,

$$\left[\frac{\mathrm{d}^{\nu}}{\mathrm{d}\,x^{\nu}}\cdot\left(\frac{\mathrm{d}^{\nu_1}}{\mathrm{d}\,x^{\nu_1}}\cdot\frac{\mathrm{d}^{\nu_2}}{\mathrm{d}\,x^{\nu_2}}\right)\right] f(x)=\left[\left(\frac{\mathrm{d}^{\nu}}{\mathrm{d}\,x^{\nu}}\cdot\frac{\mathrm{d}^{\nu_1}}{\mathrm{d}\,x^{\nu_1}}\right)\cdot\frac{\mathrm{d}^{\nu_2}}{\mathrm{d}\,x^{\nu_2}}\right] f(x)$$
$$=D^{\nu}\,.\,(D^{\nu_1}\,.\,D^{\nu_2})\,f(x)=(D^{\nu}\,.\,D^{\nu_1})\,.\,D^{\nu_2}f(x)\ , \tag{1.5}$$

and the commutative law,

$$\left(\frac{\mathrm{d}^{\nu_1}}{\mathrm{d}\,x^{\nu_1}}\cdot\frac{\mathrm{d}^{\nu_2}}{\mathrm{d}\,x^{\nu_2}}\right) f(x)=\left(\frac{\mathrm{d}^{\nu_2}}{\mathrm{d}\,x^{\nu_2}}\cdot\frac{\mathrm{d}^{\nu_1}}{\mathrm{d}\,x^{\nu_1}}\right) f(x)$$
$$=D^{\nu_1}\,D^{\nu_2}\,f(x)=D^{\nu_2}\,D^{\nu_1}\,f(x)\ , \tag{1.6}$$

of multiplication are obeyed. In other words, for ν_1 and ν_2 positive integers, the operators D^{ν_1} and D^{ν_2} commute. So far so good! But problems begin to arise if inverse powers are involved.

1.1.3 What is $D^{-1}\,f(x)$?

First thing to note here is that D represents the process of differentiation with respect to the variable x. Therefore, quite reasonably, the inverse process—meaning integration with respect to x—should be represented by D^{-1}. Simple analysis helps illustrate this point. For instance, straightforward integration of the differential equation,

$$D\,y(x)\ =\ 2\,x\ ,$$

leads to its solution

$$\int 2\,x\,.\,\mathrm{d}x\ =\ x^2+\sigma_0$$
$$=\ \int D\,y(x)\,.\,\mathrm{d}x\ =\int\frac{\mathrm{d}\,y(x)}{\mathrm{d}x}\,.\,\mathrm{d}x\ =\ \int\mathrm{d}\,y(x)\ =\ y(x) \tag{1.7}$$

where σ_0 is a constant. If we assume D^{-1} represents integration with respect to x, we would also get the same result.

$$y(x) \;=\; D^{-1}\,.\,[D\,y(x)] \;=\; D^{-1}\,.\,(2\,x) = \int (2\,x)\,.\,\mathrm{d}x \;=\; x^2 + \sigma_0 \quad .$$

Another way of looking at this is to write

$$D^{-1}\,.\,D\,y \;=\; D^{-1+1}\,y = \; D^0\,y(x) \;=\; y(x) \quad . \tag{1.8}$$

This suggests two things. First, the operator D^{-1} may be identified with the operation of integration. Second, that $D^{-1}\,.\,D \;=\; D^{-1+1}$ may be set equivalent to $D^0 \;=\; 1$.

This looks very promising. But wait a minute !

In view of (1.8), for integral ν that is $>\mu$, the following should hold:

$$\begin{aligned} D^{-\nu}\,.\,D^{\nu}\,x^{\mu} &= D^{-\nu}\,.\,0 \\ &\neq x^{\mu}\,. \end{aligned} \tag{1.9}$$

On the other hand, $D^{\nu}\,.\,D^{-\nu}\,x^{\mu}$ can be written as

$$D^{\nu}\,.\,\left[D^{-\nu}\,x^{\mu}\right] \;=\; D^{\nu}\,.\,\left[\frac{x^{\mu+\nu}}{(\mu+1)(\mu+2)\ldots(\mu+\nu)} + O\left(x^{\nu-1}\right)\right] \;=\; x^{\mu}\;. \tag{1.10}$$

The two results (1.9) and (1.10) are different. Indeed, whenever $\nu > \mu$,

$$D^{-\nu}\,.\,D^{\nu}\,x^{\mu} \neq D^{\nu}\,.\,D^{-\nu}\,x^{\mu} \quad . \tag{1.11}$$

1.1.4 Index Law

In (1.7)–(1.11), we have made use of the index law that asserts:

$$\begin{aligned} \left(\frac{\mathrm{d}^{\nu_1+\nu_2}}{\mathrm{d}\,x^{\nu_1+\nu_2}}\right) f(x) &= \left(\frac{\mathrm{d}^{\nu_1}}{\mathrm{d}\,x^{\nu_1}}\right)\,.\,\left(\frac{\mathrm{d}^{\nu_2}}{\mathrm{d}\,x^{\nu_2}}\right) f(x) \\ &= D^{\nu_1+\nu_2}\,f(x) = D^{\nu_1}\,.\,D^{\nu_2}\,f(x). \end{aligned} \tag{1.12}$$

Clearly, this law holds if both ν_1 and ν_2 are positive integers. But above we have used this law even when one of these indices was not positive? Let us try a very simple exercise to check the adequacy of the index law when that is the case.

$$D^{3-1}x^2 \;=\; D^3\,.\,D^{-1}\,x^2 = D^3.\left[\frac{x^3}{3}+\sigma_0\right] =\; 2\;, \tag{1.13}$$

but

$$D^{-1+3}x^2 \;=\; D^{-1}\,.\,[D^3\,x^2] = D^{-1}\,.\,[0] =\;\; \sigma_1. \tag{1.14}$$

Thus, an additional instruction would be needed to set $\sigma_1 = 2$. To sum up:

When n is a positive integer, D^n may be treated as an ordinary algebraic quantity. In contrast, negative powers of D are not simple algebraic quantities and must be treated with care.

Chapter 2
Some Definitions



A wealth of terminology is used in the literature. Unfortunately, some sounds similar even when it has distinctly different meaning. Noted below are a few definitions that are used in this manuscript.

2.1 Ordinary Differential Equation

A function involving a dependent variable, say u, only a single independent variable, say x, and at least one derivative of u with respect to x is an 'ordinary differential equation' (ODE).

(ODE)s are used in problems in biology, chemistry, engineering, physics, other science disciplines, as well as computational mathematics.

2.1.1 Explicit

An (ODE) may be expressed in the following explicit form

$$F_{xpl}(x, u, u^{(1)}, \ldots, u^{(\nu-1)}) = u^{(\nu)}. \tag{2.1}$$

Here, $u^{(\nu)}$ is the νth differential of the dependent variable u with respect to the independent variable x, i.e.,

$$u^{(\nu)} = \frac{\mathrm{d}^{\nu} u}{\mathrm{d}x^{\nu}} \equiv D^{\nu} u, \tag{2.2}$$

and F_{xpl} is a continuous function of the variables $x, u, u^{(1)}, \ldots, u^{(\nu-1)}$, where the independent variable x ranges within some specified interval $I \equiv X_i < x < X_f$.

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Such an equation is called an explicit ordinary differential equation. Knowledge of a function $u(x)$ defined within the same interval I that satisfies the same differential equation, i.e.,

$$F_{xpl}\left(x, u(x), u^{(1)}(x), \ldots, u^{(\nu-1)}(x)\right) = u^{(\nu)}(x), \tag{2.3}$$

is equivalent to having an explicit solution of the explicit ordinary differential equation.

2.1.2 Implicit

When a given (ODE) cannot be expressed in the explicit from, it is said to be an implicit ordinary differential equation represented as

$$F_{impl}(x, u, u^{(1)}, \ldots, u^{(\nu-1)}, u^{(\nu)}) = 0. \tag{2.4}$$

Generally, explicit ordinary differential equations are easier to treat than implicit ordinary differential equations. For simplicity, therefore, we shall work mostly with the explicit ordinary differential equations.

2.1.3 Linear

A differential equation that is of the first degree in its dependent variable as well as all its derivatives and is of the form

$$\sum_{i=0}^{\nu} a_i(x) D^i u = B(x)\ , \tag{2.5}$$

where $B(x)$ and $\{a_i(x)\}$, $i = 0, 1, \ldots, \nu$ are all known functions of x, is said to be a linear ordinary differential equation.

2.1.4 Homogeneous

Whenever the function $B(x)$ is missing from the linear (ODE), the latter is referred to as homogeneous linear ordinary differential equation. That is

$$\sum_{i=0}^{\nu} a_i(x) D^i u = 0. \tag{2.6}$$

[Unfortunately, this terminology is somewhat confusing because to many readers the normal usage of the term homogeneous function of degree-n implies the following equivalence:

$$f(\lambda x, \lambda y, \lambda z) \equiv \lambda^n f(x, y, z). \tag{2.7}$$

Later in the book, there will be occasion to work with differential equations of the type

$$F_1(\lambda x, \lambda y)\,\mathrm{d}x + F_2(\lambda x, \lambda y)\,\mathrm{d}y = \lambda^n [F_1(x, y)\,\mathrm{d}x + F_2(x, y)\,\mathrm{d}y\,, \tag{2.8}$$

where both $F_1(x, y)$ and $F_2(x, y)$ are homogeneous functions and both are of the same degree.]

2.1.5 Inhomogeneous

A linear (ODE) of the form (2.5), where $B(x)$ is not equal to zero, is called an inhomogeneous linear ordinary differential equation.

2.1.6 Nonlinear

When the (ODE) cannot be expressed in linear form represented in (2.5) or (2.6), it is said to be nonlinear (ODE). For instance, the well-known simple pendulum equation, where l represents the length, m the mass, and g the acceleration due to gravity—all of which are assumed to be constants—

$$m\frac{\mathrm{d}^2\{l\,\tan\theta\}}{\mathrm{d}t^2} + m\,g\sin\theta = 0 \tag{2.9}$$

is nonlinear when the angle θ is not very small compared to a radian. Here, θ is the angle that the massless pendulum rod of length l makes with the vertical, m is the mass hanging at the bottom of the rod, t refers to the time, g is the acceleration due to gravity, and $\sqrt{\frac{g}{l}}$ is the angular velocity so that one complete cycle—which is equivalent to angular rotation 2π—is completed in time $\frac{2\pi}{\sqrt{\frac{g}{l}}}$. It does, however, become linear for very small θ when both $\tan\theta$ and $\sin\theta$ tend to θ.

Just as implicit (ODE) is harder to solve than explicit (ODE), solving nonlinear (ODE) requires more effort than solving linear (ODE). Furthermore, simple treatment of nonlinear (ODE) cannot be guaranteed to succeed. Fortunately, many of the physical problems of interest—at least in the first approximation—can be expressed in terms of explicit ordinary differential equations that are linear. Therefore, much of our attention will be focused on linear explicit ordinary differential equations.

2.1.7 Partial Differential Equation

An (ODE) has only one independent variable. But physical situations of interest sometime involve more than one physical property. As a result, the functions representing them may sometime involve two or more independent variables. And such dependence is often best expressed by partial differential equations that contain partial derivatives with respect to more than one independent variable.

2.1.8 The Order of an (ODE)

The order of an (ODE) is ν if the highest derivative in the equation is of order ν. For instance, this is the case in (2.1), (2.3), and (2.4) because the highest derivative there is $u^{(\nu)}$.

Similarly, the order of a partial (ODE) is the order of the highest partial derivative present.

2.1.9 The Degree of an (ODE)

Unlike the order of an (ODE), determining its degree requires some care. It is necessary first to arrange the (ODE) in a form where all the differential coefficients in the differential equation appear with rational and integral powers. When that has been done, the equation can be written as a polynomial in all the derivatives present. Then, the power of the derivative of the highest order is the degree of the given (ODE).

2.1.10 Order and Degree: Exercises

Determine the order and degree of the following differential equations.

$$\left(D^3u\right)^2 = A(x)\,D^2u + B(x)\,Du + G(x)\ , \tag{2.10}$$

$$\left(D^2u\right)^{\frac{1}{\sqrt{2}}} = u + A(x)\ , \tag{2.11}$$

$$\left(D^2u\right)^{\frac{2}{3}} = \left[1 + \left(D^3u\right)^{\frac{1}{2}}\right]^2\ , \tag{2.12}$$

$$\left(\frac{\partial^2 u}{\partial^2 y}\right)_x = A(x,y)\left(\frac{\partial^2 u}{\partial^2 x}\right)_y . \tag{2.13}$$

The highest order derivative in (2.10) is D^3u. Therefore, it is of third-order. Also, the differentials that appear here have powers that are integral. And, the highest order differential occurs with a power of 2. Therefore, (ODE) (2.10) is of second degree.

Regarding (ODE) (2.11), the only and therefore the highest order derivative is D^2u. As such, this (ODE) is of second order. To determine its degree, one needs first to rationalize all its derivatives. To this end, raise both sides to the power $\sqrt{2}$.

$$D^2u = [(u + A(x)]^{\sqrt{2}} . \tag{2.14}$$

The highest-order derivative is clearly of power unity. Therefore, (ODE) (2.11) is of first degree.

For (2.12), proceed as follows: Raise both sides to power $\frac{3}{2}$.

$$D^2u = \left[1 + (D^3u)^{\frac{1}{2}}\right]^3 \;=\; 1 + 3\,D^3u + 3\left(D^3u\right)^{\frac{1}{2}} + \left(D^3u\right)^{\frac{3}{2}} .$$

Now transfer $1 + 3\,D^3u$ to the left-hand side

$$\left[D^2u - 3\,D^3u - 1\right] = (D^3u)^{\frac{1}{2}}\left[3 + D^3u\right]$$

and square both sides.

$$\left[D^2u - 3\,D^3u - 1\right]^2 = \left(D^3u\right)\left[3 + D^3u\right]^2 \quad . \tag{2.15}$$

The highest-order differential in (2.15) is D^3u. Also, the highest power it occurs with—see the right-hand side—is $\left(D^3u\right)^3$. Therefore, the (ODE) (2.11) is of third order and third degree.

The partial differential equation (2.13) is clearly of second order and first degree.

2.1.11 Characteristic Equation: E_{ch}

When $\exp(kx)$ is tried as a solution to an nth order homogeneous linear equation, the result appears in the form $\exp(kx)$ multiplied by an nth degree polynomial in the variable k. Such polynomial will be referred to as the characteristic equation of the given nth order homogeneous linear ordinary differential equation.

2.1.12 Complementary Solution: S_{comp}

Determine the n solutions of the nth degree polynomial in k—that is, contained in the characteristic equation mentioned above —and notate them k_j with $j = 1, 2, \ldots, n$. Next, exponentiate each one of them. Then, the sum

$$\sum_{j=1}^{n} \sigma_j \exp k_j \ , \tag{2.16}$$

where σ_j are arbitrary constants, is called a complementary solution.

2.1.13 Particular Integral: I_{pi}

A solution of an inhomogeneous linear (ODE) that contains no arbitrary constants will be called its particular integral I_{pi}.

2.1.14 Indicial Equation

The term with the lowest power of x—such as (11.76)—in a Frobenius solution—such as (11.75)—is termed the indicial equation.

2.1.15 General Solution

A solution that contains arbitrary constants is often called a general solution. General solution is a bit of a misnomer because the requirement for a solution to be general is quite relaxed. Of the possible n arbitrary constants in the solution, only one needs be nonzero. Occasionally in this text, general solution will refer to a solution given in terms of the variable ν_0 that symbolizes roots of the indicial equation.

2.1.16 Complete Solution

Assume the n functions in the complementary solution—namely, $\sigma_j \exp k_j$—every one of which is itself a solution of the nth order linear (ODE) are linearly independent. Add the complementary solution to the particular integral I_{pi}. The sum of the two makes up a complete solution. That is,

$$Complete\ Solution = I_{pi} + Complementary\ Solution\,. \tag{2.17}$$

[Note: Linear independence is explained at length in (3.43)–(3.47). Also note that by making appropriate adjustment of the n arbitrary multiplying constants,

$\sigma_i,\ i = 1, \ldots, n$, derived from the complete solution all possible solutions can be obtained. Accordingly, complete solution may have many equivalent forms.]

2.1.17 Complete Primitive

Should all the n arbitrary multiplying constants, $\sigma_i,\ i = 1, \ldots, n$, that occur in the complementary solution stand determined by the use of appropriate boundary constants then the resultant complete solution is called complete primitive.

2.1.18 Singular Solution

Occasionally, there may exist a solution to a given differential equation that cannot be derived from its complete solution or complete primitive. Such an unusual solution is often called a singular solution.

2.2 How Some (ODE) Arise

Upon Elimination of Constants

While an appropriate physical motivation is the source of most differential equations, differential equations may also arise in other ways. In particular, they may result from elimination of unknown constants in an equation. For instance, upon differentiation the following equations with one and two constants may be expressed as first- and second-order differential equations, respectively. In this manner, the equations

$$\begin{aligned} u(x) &= \sigma_0 \exp(k\,x) \quad , \\ v(x) &= \sigma_1 \sin(a\,x) + \sigma_2 \cos(a\,x) \quad , \end{aligned} \tag{2.18}$$

would lead to the following differential equations.

$$\begin{aligned} Du &= k\,u \quad ; \\ D^2 v &= -a^2\,v \quad . \end{aligned} \tag{2.19}$$

Notice that by eliminating a single constant, e.g., σ_0, a first-order (ODE) is derived while a differential equation of second order is obtained from eliminating two constants such as σ_1 and σ_2. Thus, it is usually the case that the elimination of n constants leads to a differential equation of the nth order. Conversely, solution to a differential equation of the nth order usually contains n unknown constants.

As an aside each of the surviving known constants k and a in (2.19) can also be eliminated by further differentiation which raises the order of each of the two differential equations by one notch.

$$\begin{aligned} u\, D^2 u &= (Du)^2 \ , \\ v\, D^3 v &= (D^2 v)(Dv) \ . \end{aligned} \tag{2.20}$$

Chapter 3
Constant Coefficients



Most differential equations that describe vibrational motion belong to a class of linear ordinary differential equations where the coefficients are known constants. User-friendly methods for solving both the homogeneous and the inhomogeneous versions of these equations are described in this chapter.

3.1 Homogeneous Linear (ODEs)

Homogeneous linear ordinary differential equations with constant coefficients are of the form

$$\sum_{i=0}^{\nu} c_i \; D^i u(x) = 0 \tag{3.1}$$

where c_i for $i = 0, 1, 2, \ldots$, etc., are known constants. Exact solution of such equations can often be found in terms of elementary functions when the equations are first order or second order, and also—with somewhat greater effort—when they are third order or fourth order.

3.1.1 First Order

The simplest homogeneous linear ordinary differential equation is of the first order.

$$\sum_{i=0}^{1} c_i \; D^i u(x) = (c_0 + c_1 \; D)\, u(x) = 0 \; .$$

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Rewrite the above equation

$$\frac{\mathrm{d}u(x)}{u(x)} = \left(-\frac{c_0}{c_1}\right)\mathrm{d}x \tag{3.2}$$

and integrate.

$$\ln u(x) = \left(-\frac{c_0}{c_1}\right) x + \sigma \ . \tag{3.3}$$

The solution is

$$u(x) = \sigma_0 \exp\left[\left(-\frac{c_0}{c_1}\right) x\right] \ , \tag{3.4}$$

where $\sigma_0 = \exp(\sigma)$ is the single unknown arbitrary constant that can be determined from one boundary condition. For instance, the value of $u(x)$ at $x = 0$ is equal to σ_0.

Notice that only one integration is needed to obtain a solution of the first-order homogeneous linear ordinary differential equation with constant coefficients and the solution has only a single arbitrary constant. Indeed, the number of independent constants that a complete solution of an nth order homogeneous linear ordinary differential equation with constant coefficients contains usually is equal to n and such a solution requires the equivalent of n integrations.

3.1.2 Second Order

The following is a second-order homogeneous linear ordinary differential equation with constant coefficients.

$$\left[c_0 + c_1\ D + c_2\ D^2\right] u(x) = 0 \ . \tag{3.5}$$

Given our experience with a first-order homogeneous linear ordinary differential equation with constant coefficients—see (3.4)—let us again try

$$u(x) = \exp(k\,x) \tag{3.6}$$

as a possible solution. With this choice, (3.5) becomes

$$\begin{aligned}&\left[c_0 + c_1\ D + c_2\ D^2\right] \exp(k\,x) = \\ &\quad \exp(k\,x)\left[c_0 + c_1\,k + c_2\,k^2\right] = 0 \ .\end{aligned} \tag{3.7}$$

Except for the extraordinary circumstance when $k\,x \to -\infty$, (3.7) requires

$$\left[c_0 + c_1\,k + c_2\,k^2\right] = 0 \quad . \tag{3.8}$$

This is the characteristic equation. Note, given a differential equation, e.g. (3.5), the characteristic equation is readily found by replacing D with k.

3.1.3 Characteristic Equation: E_{ch}

The characteristic equation for all second-order homogeneous linear ordinary differential equation is a quadratic. The two roots of (3.8) are k_1 and k_2.

$$k_1 \;=\; (2\,c_0)^{-1}\left[-c_1 + \sqrt{c_1^2 - 4\,c_0\,c_2}\right] \quad , \tag{3.9}$$

$$k_2 \;=\; (2\,c_0)^{-1}\left[-c_1 - \sqrt{c_1^2 - 4\,c_0\,c_2}\right] \quad . \tag{3.10}$$

3.1.4 Unequal Roots

Assuming the two roots, whether they be real or imaginary, are unequal, then the differential equation (3.5) has two solutions of the form

$$u(x) \;=\; u_1(x) = \exp\left(k_1\,x\right) \quad . \tag{3.11}$$

$$u(x) \;=\; u_2(x) = \exp\left(k_2\,x\right) \quad . \tag{3.12}$$

3.1.5 Complementary Solution

And the complementary solution is their linear sum. That is

$$S_{\text{comp}} = \sigma_1\,\exp\left(k_1\,x\right) + \sigma_2\,\exp\left(k_2\,x\right) \quad . \tag{3.13}$$

As usual, σ_1 and σ_2 represent arbitrary constants.

Superposition Principle

When $\exp\left(k_1\,x\right)$ and $\exp\left(k_2\,x\right)$ are solutions of a second-order homogeneous linear ordinary differential equation with constant coefficients, then a linear sum of the two, namely the complementary solution,

$$\sigma_1 \exp(k_1 x) + \sigma_2 \exp(k_2 x) \ , \tag{3.14}$$

is also a solution of the same homogeneous linear ordinary differential equation. Thus, the superposition principle asserts that the following must be true:

$$\left[c_0 + c_1 D + c_2 D^2\right].[\sigma_1 \exp(k_1 x) + \sigma_2 \exp(k_2 x)] = 0. \tag{3.15}$$

Knowing that a given set of derivatives of the product $\sigma u(x)$ is equal to σ times the same set of derivatives of $u(x)$, and the fact that derivatives of a sum are equal to the sum of those derivatives, (3.15) can be rewritten as

$$\begin{aligned} &\sigma_1 \left[c_0 + c_1 D + c_2 D^2\right] u_1(x) \\ &+\sigma_2 \left[c_0 + c_1 D + c_2 D^2\right] u_2(x) = 0 \ . \end{aligned} \tag{3.16}$$

According to (3.9)–(3.13), the relationship (3.16) is indeed satisfied. This confirms the assertion (3.15) and therefore the superposition principle.

3.1.6 Examples Group I: Unequal Real Roots

Complementary solution is worked out for three differential equations for which the characteristic equation as well as its roots is provided. Note, these are second-order homogeneous linear ordinary differential equation with constant coefficients, whose characteristic equation has two real roots, k_1 and k_2, that are unequal.

$$\begin{aligned} &\left[D^2 + 4D - \frac{9}{4}\right] u = 0 \ ; \ \ E_{ch} \ = \ k^2 + 4k - \frac{9}{4} \ = \ 0 \ , \\ &k_1 \ = \ \frac{1}{2} \ , \ \ k_2 \ = \ -\frac{9}{2} \ . \end{aligned} \tag{3.17}$$

$$\begin{aligned} &\left[2D^2 + 5D + 3\right] u = 0 \ ; \ \ E_{ch} \ = \ 2k^2 + 5k + 3 \ = \ 0 \ , \\ &k_1 \ = \ -1 \ , \ \ k_2 \ = \ -\frac{3}{2} \ . \end{aligned} \tag{3.18}$$

$$\begin{aligned} &\left[2D^2 - 5D + 3\right] u = 0 \ ; \ \ E_{ch} \ = \ 2k^2 - 5k + 3 \ = \ 0 \ , \\ &k_1 \ = \ 1 \ , \ \ k_2 \ = \ \frac{3}{2} \ . \end{aligned} \tag{3.19}$$

As described in (3.5) → (3.13), the relevant complementary solution for the differential equations in (3.17)–(3.19) is the following.

$$S_{\text{comp}} \ = \ \sigma_1 \exp(k_1 x) + \sigma_2 \exp(k_2 x) \ . \tag{3.20}$$

3.1.7 Examples Group II : Complex Roots

Treated below is a second-order homogeneous linear ordinary differential equation with constant coefficients whose characteristic equation is a quadratic with complex roots k_1 and k_2.

$$k_1 \;=\; r - i\,m \;\;;\;\; k_2 \;=\; r + i\,m \;\;. \tag{3.21}$$

According to (3.20), the complementary equation takes the form

$$\begin{aligned} S_{\text{comp}} &= \sigma_1 \,\exp{(r - i\,m)\,x} + \sigma_2 \,\exp{(r + i\,m)x} \\ &= \exp{(r\,x)}\left[\sigma_1 \,\exp{(-i\,m\,x)} + \sigma_2 \,\exp{(i\,m\,x)}\right] \\ &= \exp{(r\,x)}\left[\sigma_3 \,\sin(m\,x) + \sigma_4 \,\cos(m\,x)\right] \;\;, \end{aligned} \tag{3.22}$$

where the arbitrary constants σ_3 and σ_4 represent combinations of the earlier arbitrary constants in the form

$$\begin{aligned} \sigma_3 &= -i\,(\sigma_1 - \sigma_2) \;\;, \\ \sigma_4 &= \quad\;\; (\sigma_1 + \sigma_2) \;\;. \end{aligned} \tag{3.23}$$

In deriving the above equation, the following identity was also used

$$\exp{(\pm i\,\theta)} = \cos(\theta) \pm i\,\sin(\theta) \;\;. \tag{3.24}$$

3.1.8 Equations with Complex Roots

Given below are differential equations along with their characteristic equations and complex roots in the form of r and m. [Note: See (3.21) for the definition of r and m.]

$$\begin{gathered} \left[D^2 + 2D + 5\right]u(x) = 0 \;\;;\;\; k^2 + 2k + 5 \;=\; 0 \;\;; \\ r \;=\; -1 \;\;,\;\; m \;=\; 2 \;\;. \end{gathered} \tag{3.25}$$

$$\begin{gathered} \left[D^2 - 6D + 10\right]u(x) = 0 \;\;;\;\; k^2 - 6k + 10 \;=\; 0 \;\;; \\ r \;=\; 3 \;\;,\;\; m \;=\; 1 \;\;. \end{gathered} \tag{3.26}$$

$$\begin{gathered} \left[D^2 + 3D + \frac{5}{2}\right]u(x) = 0 \;\;;\;\; k^2 + 3k + \frac{5}{2} \;=\; 0 \;\;; \\ r \;=\; -\frac{3}{2} \;\;,\;\; m = \frac{1}{2} \;\;. \end{gathered} \tag{3.27}$$

The relevant complementary solution is obtained by introducing these values of r and m into (3.22).

3.1.9 Equation with Double Root

Consider next a second-order homogeneous linear ordinary differential equation with constant coefficients whose characteristic equation is a quadratic with a double root, real or imaginary. For instance, when the differential equation

$$\left[D^2 - 2\,c\,D + c^2\right] u(x) = (D-c)^2\, u(x) \;= 0 \tag{3.28}$$

is solved by setting $u(x) = \exp(k\,x)$, its E_{ch}, namely

$$(k-c)^2 \;= 0 \;, \tag{3.29}$$

yields identical two roots $k = c$. Therefore,

$$u(x) = u_1(x) = \exp(c\,x) \tag{3.30}$$

is a double solution that duly satisfies the differential equation (3.28). But it is only a single distinct solution. [Note: Two solutions are distinct if they are linearly independent. Linear independence is formally defined in (3.43)–(3.45).] Hoping that another distinct solution would also involve $\exp(c\,x)$, let us try

$$u(x) \;=\; u_2(x) = \exp(c\,x)\, f(x) \tag{3.31}$$

as a possible second distinct solution and determine $f(x)$ accordingly. To this end, insert (3.31) into (3.28). One gets

$$\begin{aligned} 0 \;=\; (D-c)^2\, u_2(x) &= (D-c)^2 \left[\exp(c\,x)\, f(x)\right] \\ = (D-c)\left\{(D-c)\left[\exp(c\,x)\, f(x)\right]\right\} &= (D-c)\left\{\exp(c\,x)\; Df(x)\right\} \\ &= \exp(c\,x)\left[D^2\, f(x)\right] \;. \end{aligned} \tag{3.32}$$

Thus, all that is needed is to find the solution to

$$\left[D^2\, f(x)\right] = \frac{\mathrm{d}}{\mathrm{d}x}\left\{Df(x)\right\} = 0 \;. \tag{3.33}$$

This is readily done by integrating (3.33) twice. The first integration leads to

$$\begin{aligned} \int \left[D^2\, f(x)\right] \mathrm{d}x \equiv \int \left[\frac{\mathrm{d}}{\mathrm{d}x}\left\{Df(x)\right\}\right] \mathrm{d}x &= \int [0] \mathrm{d}x \;, \\ = \left\{Df(x)\right\} + \mathrm{const}_1 &= \mathrm{const}_2 \;. \end{aligned} \tag{3.34}$$

And the second to

$$\int [D\, f(x)]\, \mathrm{d}x = (\mathrm{const}_2 - \mathrm{const}_1) \int \mathrm{d}x \ ,$$
$$f(x) + \mathrm{const}_3 = (\mathrm{const}_2 - \mathrm{const}_1)\,(x + \mathrm{const}_4) \ . \tag{3.35}$$

By setting $(\mathrm{const}_2 - \mathrm{const}_1) = \sigma_1$ and $(\sigma_1 \,.\, \mathrm{const}_4 - \mathrm{const}_3) = \sigma_0$, one gets

$$f(x) = \sigma_0 + \sigma_1\, x \ . \tag{3.36}$$

Thus, according to (3.31) and (3.36), the $u(x)$ given below solves the differential equation $(D - c)^2\, u(x) = 0$.

$$u(x) = \exp(c\, x)\, f(x) = \exp(c\, x)\,(\sigma_0 + \sigma_1\, x) \ . \tag{3.37}$$

This version of $u(x)$—which is to be dubbed the solution—is quite interesting. In addition to including a new expression, namely $\exp(c\, x)\,(\sigma_1\, x)$, it contains also all of the first solution, namely $\exp(c\, x)\,(\sigma_0)$, that was given in (3.30). Additionally, because $u(x)$ has been derived from two integrations, it has the requisite number of unknown constants: namely σ_0 and σ_1. Therefore, $u(x)$, given in (3.37), is the complementary solution. Finally, as long as its two parts—namely $(\sigma_0)\, \exp(c\, x)$ and $(\sigma_1\, x)\, \exp(c\, x)$—are mutually distinct, meaning they are linearly independent, $u(x)$ is also the complete solution of the differential equation (3.28). Additionally, if the two unknown constants, σ_0 and σ_1, are determined—say, from two boundary conditions—this solution would also qualify as the complete primitive. [Note: See the succeeding subsection for a discussion of when a given set of functions is mutually distinct, meaning when they are linearly independent. Also see (3.47)–(3.53) where the linear independence of given two expressions is demonstrated.]

3.1.10 n-Equal Roots

The foregoing can be extended to nth-order homogeneous linear ordinary differential equation with constant coefficients.

$$(D - c)^n\, u(x) = 0 \ .$$

Clearly, its characteristic equation

$$(k - c)^n = 0$$

has n equal roots: $k = c$. Following the same procedure as outlined in deriving (3.37), the complementary solution is

$$S_{\text{comp}}(x) \;= \exp(c\,x)[\sigma_0 + \sigma_1\,x + \sigma_2\,x^2 + \cdots + \sigma_{n-1}\,x^{n-1}]. \tag{3.38}$$

And once again it is a sum of distinct terms. As such, it is a complete solution. If the values of $\sigma_i - i = 0\,,\ 1\,,\ 2\,, \ldots,\ (n-1)$ were all known—say, determined from n different boundary conditions—the $S_{\text{comp}}(x)$, given above, would also qualify as the complete primitive.

Characteristic equations with multiple roots are analyzed below.

Equation (3.38) refers to the case where E_{ch} has n equal roots. That means, there is only one distinct root that occurs n different times. What if there were two roots, both of them occurring multiple times? To study this case, consider the following differential equation with two roots: c_0 and c.

$$F\,u(x) \;=\; (D - c_0)^{n_0}\,(D - c)^n\;u(x) \;=\; (D - c)^n\;(D - c_0)^{n_0}\;u(x) \;=\; 0\;. \tag{3.39}$$

3.1.11 E_{ch} with Multiple Roots

Two Roots Equal and Occurring Multiple Times

If the two roots c_0 and c are equal, the problem (3.39) is similar to that worked out in (3.38). Then, the result is

$$u(x) = S_{\text{comp}}(x) = \exp(c\,x)\left(\sigma_0 + \sigma_1\,x + \sigma_2\,x^2\;+ \cdots + \sigma_{n_0+n-1}x^{n_0+n-1}\right) \tag{3.40}$$

Two Roots, Unequal and Occurring Multiple Times

On the other hand, for the more interesting case, should c_0 and c be unequal, the above differential equation would have two distinct roots: namely c_0, that occurs n_0 times, and c, that occurs n times. The characteristic equation then would be

$$E_{ch} \;=\; (k - c_0)^{n_0}\,(k - c)^n \;=\; (k - c)^n\,(k - c_0)^{n_0}\;\;.$$

Define

$$(D - c_0)^{n_0}\,U_0(x) = 0\,,$$
$$(D - c)^n\,U(x) = 0\,.$$

Then,

$$\begin{aligned}
&(D-c_0)^{n_0}\,(D-c)^n\,[U(x)+U_0(x)]\\
&=(D-c_0)^{n_0}\,(D-c)^n\,U(x)+(D-c_0)^n\,(D-c)^n\,U_0(x)\\
&=0+(D-c_0)^{n_0}\,(D-c)^n\,U_0(x)=0+(D-c)^n\,(D-c_0)^{n_0}\,U_0(x)\\
&=0+0\ . \qquad (3.41)
\end{aligned}$$

Thus, the solution of homogeneous linear ordinary differential equation with constant coefficients (3.39) is

$$\begin{aligned}
S_{\text{comp}}(x)=u(x)\ &=\ U_0(x)+U(x)\\
&=\exp(c_0\,x)\left[\sigma_{00}+\sigma_{01}\,x+\sigma_{02}\,x^2\ +\cdots+\sigma_{0(n_0-1)}x^{n_0-1}\right]\\
&+\exp(c\,x)\left[\sigma_0+\sigma_1\,x+\sigma_2\,x^2\ +\cdots+\sigma_{n-1}x^{n-1}\right]\ . \qquad (3.42)
\end{aligned}$$

3.1.12 Problems Group I

Find complementary solution, $S_{\text{comp}}(x)$, to the following ten homogeneous linear ordinary differential equations with constant coefficients.

$$\begin{aligned}
\left(D^2+2D-3\right)u(x)&=0\ .\ (1)\\
\left(D^2-3D-4\right)u(x)&=0\ .\ (2)\\
\left(D+\frac{1}{2}\right)^2u(x)&=0\ .\ (3)\\
\left(D-\frac{1}{2}\right)^2u(x)&=0\ .\ (4)\\
(D+2)^3\,u(x)&=0\ .\ (5)\\
(D-2)^3\,u(x)&=0\ .\ (6)\\
\left(D^2+D+1\right)u(x)&=0\ .\ (7)\\
\left(D^2-D+1\right)u(x)&=0\ .\ (8)\\
\left(D^2+2D+3\right)u(x)&=0\ .\ (9)\\
\left(D^2+3D+4\right)u(x)&=0\ .\ (10)
\end{aligned}$$

3.2 Linear Dependence and Linear Independence

Linear Dependence

A given set of functions $\{f_i(x)\}$, $i=1,2,...,n$, is said to be 'linearly dependent' if and only if, for all values of x in an interval I, there exist n constants, say $\{\sigma_i\}$,

$i = 1, 2, ..., n$, such that the following relationship—namely (3.43)—holds true.

$$\sum_{i=1}^{n} \sigma_i f_i(x) = 0 \ . \tag{3.43}$$

Clearly, the case when the constants $\{\sigma_i\}$, $i = 1, 2, ..., n$, are all vanishing is trivial.

A straightforward use of the relationship (3.43) for determining linear dependence of the functions $f_i(x)$ is quite awkward. It requires knowledge of an appropriate, non-trivial, set of constants σ_i. This requirement is circumvented below.

In order to determine whether the given n functions, $\{f_i(x)\}, i = 1, 2, ..., n$—each of which is differentiable $(n-1)$ times—are linearly dependent, one differentiates (3.43) several times and keeps a record. Every time (3.43) is differentiated, the process produces a new differential equation. This way, after $(n-1)$ differentiations, there are $(n-1)$ new differential equations. These along with the original equation, namely (3.43), make a total of n simultaneous homogeneous linear ordinary differential equations with constant coefficients that involve the n constants $\{\sigma_i\}$, $i = 1, 2, ..., n$. It is helpful to display these equations.

$$\begin{aligned}
\sum_{i=1}^{n} \sigma_i f_i(x) &= 0 \ ; \quad (1) \\
\sum_{i=1}^{n} \sigma_i \, . \, Df_i(x) &= 0 \ ; \quad (2) \\
&\ldots\ldots \ . \ \ldots\ldots \\
\sum_{i=1}^{n} \sigma_i \, . \, D^{n-1} f_i(x) &= 0 \ ; \quad (n)
\end{aligned} \tag{3.44}$$

For the given n functions, $\{f_i(x)\}$, $i = 1, 2, ..., n$—each of which is differentiable $(n-1)$ times—to be linearly dependent, it is required that (3.44) be satisfied by a non-trivial choice of the constant σ_i. To ensure that it is so, proceed as follows.

3.2.1 Wronskian

A well-known theorem of algebra states that n simultaneous homogeneous linear equations involving n constants—such as σ_i's, i = 1, 2, ..., n, in (3.44)—have a non-trivial solution if and only if the following relationship—that is, (3.45) given below—holds true.

$$W(x) \equiv \begin{vmatrix} f_1(x) & f_2(x), ..., f_n(x) \\ Df_1(x) & Df_2(x), ..., Df_n(x) \\ ..... & ..... \\ D^{n-1}f_1(x) & D^{n-1}f_2(x), ..., D^{n-1}f_n(x) \end{vmatrix} = 0 \,. \tag{3.45}$$

In other words:

For all values of x within an interval I, the given n functions—$\{f_i(x)\}$, $i = 1, 2, ..., n$—each of which is differentiable at least $(n-1)$ times, are linearly dependent if and only if their Wronskian, $W(x)$, is vanishing.

Linear Independence

On the other hand, linear independence, being the opposite of linear dependence, obtains only if a relationship like (3.43) never holds true—except, of course, for the trivial case when all the constants σ_i are zero. Indeed, a given set of functions $\{f_i(x)\}, i = 1, 2, ..., n$, is said to be linearly independent if and only if, for all values of x in an interval I, there exist n non-trivial constants, say $\{\sigma_i\}, i = 1, 2, ..., n$, such that relationship (3.46) obtains.

$$\sum_{i=1}^{n} \sigma_i f_i(x) \neq 0 \,. \tag{3.46}$$

Employing the same argument that led from (3.43) to (3.45), one concludes that functions $\{f_i(x)\}, i = 1, 2, ..., n$, are linearly independent if and only if the following relationship—namely (3.47)—holds true.

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x), ..., f_n(x) \\ Df_1(x) & Df_2(x), ..., Df_n(x) \\ ..... & ..... \\ D^{n-1}f_1(x) & D^{n-1}f_2(x), ..., D^{n-1}f_n(x) \end{vmatrix} \neq 0 \,. \tag{3.47}$$

Thus to reiterate: For all values of x within an interval I, functions $\{f_i(x)\}$, $i = 1, 2, ..., n$, each of which is differentiable at least $(n-1)$ times, are linearly independent if and only if their Wronskian, $W(x)$, is non-vanishing.

3.2.2 Examples Group III

(A): Work out the requirement that must be satisfied for a given pair of functions, $f_1(x)$ and $f_2(x)$, each of which is differentiable at least once, to be linearly dependent.

Solution

According to (3.45), two such functions $f_1(x)$ and $f_2(x)$ are linearly dependent if and only if they obey the relationship

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) \\ Df_1(x) & Df_2(x) \end{vmatrix} = 0 \,. \tag{3.48}$$

Rewriting (3.48),

$$\frac{Df_2(x)}{f_2(x)} - \frac{Df_1(x)}{f_1(x)} = 0 \,, \tag{3.49}$$

and integrating,

$$\log[f_2(x)] = \log[f_1(x)] + \text{constant} \,, \tag{3.50}$$

gives

$$f_2(x) = C_0 f_1(x) \,. \tag{3.51}$$

Simply expressed, two functions that are proportional are linearly dependent.

(B): Given $u(x)$ in (3.37) is complementary solution of the homogeneous linear ordinary differential equation with constant coefficients (3.28) that has equal roots, show that $u(x)$ is also its complete solution $S_{cs}(x)$.

In order for (3.37) to be a complete solution of the homogeneous linear ordinary differential equation with constant coefficients (3.28), the given two functions $f_1(x) = \exp(c\,x)$ and $f_2(x) = x\,\exp(c\,x)$ have to be linearly independent. This will be the case, according to (3.47), if the following holds true.

$$\frac{Df_2(x)}{f_2(x)} - \frac{Df_1(x)}{f_1(x)} \neq 0 \,. \tag{3.52}$$

Equation (3.52) translates into

$$\left\{ \frac{\exp(c\,x) + c\,x\,\exp(c\,x)}{x\,\exp(c\,x)} \right\} - \frac{c\,\exp(c\,x)}{\exp(c\,x)} \neq 0 \tag{3.53}$$

which reduces to the inequality

$$\frac{1}{x} \neq 0 \,. \tag{3.54}$$

Clearly, as long as x is non-infinite this inequality holds. Therefore, the given two functions are linearly independent, and as a result (3.37) is a complete solution of the homogeneous linear ordinary differential equation with constant coefficients (3.28).

3.2.3 Examples Group IV

Using the methods described in the foregoing, work out complementary solution to the set of twelve homogeneous linear ordinary differential equations with constant coefficients given in (3.55) below.

$$
\begin{aligned}
\left(D^2+3D+1\right)u(x) &= 0\ . && (1)\\
\left(D^2+3D-1\right)u(x) &= 0\ . && (2)\\
\left(D^2+3D-3\right)u(x) &= 0\ . && (3)\\
\left(D^2+3D+\frac{9}{2}\right)u(x) &= 0\ . && (4)\\
\left(D^2+4D+6\right)u(x) &= 0\ . && (5)\\
\left(D^2+D+\frac{1}{2}\right)u(x) &= 0\ . && (6)\\
\left(D+\frac{3}{2}\right)^2 u(x) &= 0\ . && (7)\\
\left(D-\frac{3}{2}\right)^2 u(x) &= 0\ . && (8)\\
(D+2)^2\,u(x) &= 0\ . && (9)\\
(D+1)^3\,u(x) &= 0\ . && (10)\\
\left[D\,(D^2-4)^2\right]u(x) &= 0\ . && (11)\\
\left[D^2\,(D^2-4)\right]u(x) &= 0\ . && (12)
\end{aligned}
\qquad (3.55)
$$

Solutions to Examples Group IV, (3.55)

In the following, complementary solution is worked out for all of the twelve homogeneous linear ordinary differential equations with constant coefficients given above in (3.55). This is done by determining their characteristic equation and its roots. Remember, as always, the characteristic equation is determined by replacing D by k.

$$
\begin{aligned}
&(1)\ \ E_{ch}\ :\ k^2+3k+1\ =\ 0\ ;\ \ k_{1,2}\ =-\frac{3}{2}\pm\frac{\sqrt{5}}{2}\ ;\\
&(1)\ \ S_{\text{comp}}(x)=\exp\left(-\frac{3}{2}x\right)\left[\sigma_1\exp\left(\frac{\sqrt{5}}{2}x\right)+\sigma_2\exp\left(-\frac{\sqrt{5}}{2}x\right)\right].\\
&(2)\ \ E_{ch}\ :\ k^2+3k-1\ =\ 0\ ;\ \ k_{1,2}\ =\ -\frac{3}{2}\pm\frac{\sqrt{13}}{2}\ ;\\
&(2)\ \ S_{\text{comp}}(x)=\exp\left(-\frac{3}{2}x\right)\left[\sigma_1\exp\left(\frac{\sqrt{13}}{2}x\right)+\sigma_2\exp\left(-\frac{\sqrt{13}}{2}x\right)\right].
\end{aligned}
$$

$$(3)\ E_{ch}\ :\ k^2+3k-3\ =\ 0\ ;\ \ k_{1,2}\ =\ -\frac{3}{2}\pm\frac{\sqrt{21}}{2}\ ;$$

$$(3)\ S_{\rm comp}(x)=\exp\left(-\frac{3}{2}x\right)\left[\sigma_1\exp\left(\frac{\sqrt{21}}{2}x\right)+\sigma_2\exp\left(-\frac{\sqrt{21}}{2}x\right)\right].$$

$$(4)\ E_{ch}\ :\ k^2+3k+\frac{9}{2}\ =\ 0\ ;\ \ k_{1,2}\ =\ -\frac{3}{2}\pm i\frac{3}{2}\ ;$$

$$(4)\ S_{\rm comp}(x)=\exp\left(-\frac{3}{2}x\right)\left[\sigma_1\sin\left(\frac{3}{2}x\right)+\sigma_2\cos\left(\frac{3}{2}x\right)\right].$$

$$(5)\ E_{ch}\ :\ k^2+4k+6\ =\ 0\ ;\ \ k_{1,2}\ =\ -2\pm i\sqrt{2}\ ;$$

$$(5)\ S_{\rm comp}(x)=\exp(-2x)\left[\sigma_1\sin\left(x\sqrt{2}\right)+\sigma_2\cos\left(x\sqrt{2}\right)\right].$$

$$(6)\ E_{ch}\ :\ k^2+k+\frac{1}{2}\ =\ 0\ ;\ \ k_{1,2}\ =\ -\frac{1}{2}\pm i\frac{1}{2}\ ;$$

$$(6)\ S_{\rm comp}(x)=\exp\left(-\frac{x}{2}\right)\left[\sigma_1\sin\left(\frac{x}{2}\right)+\sigma_2\cos\left(\frac{x}{2}\right)\right].$$

$$(7)\ E_{ch}\ :\ \left(k+\frac{3}{2}\right)^2\ =\ 0\ ;\ \ k_{1,2}\ =\ -\frac{3}{2}\ ;$$

$$(7)\ S_{\rm comp}(x)=\exp\left(-\frac{3}{2}x\right)(\sigma_0+\sigma_1x)\,.$$

$$(8)\ E_{ch}\ :\ \left(k-\frac{3}{2}\right)^2\ =\ 0\ ;\ \ k_{1,2}\ =\ \frac{3}{2}\ ;$$

$$(8)\ S_{\rm comp}(x)=\exp\left(\frac{3}{2}x\right)(\sigma_0+\sigma_1x)\,.$$

$$(9)\ E_{ch}\ :\ (k+2)^2\ =\ 0\ ;\ \ k_{1,2}\ =\ -2\ ;$$

$$(9)\ S_{\rm comp}(x)=\exp(-2x)\,(\sigma_0+\sigma_1x)\,.$$

$$(10)\ E_{ch}\ :\ (k+1)^3\ =\ 0\ ;\ \ k_{1,2,3}\ =\ -1\ ;$$

$$(10)\ S_{\rm comp}(x)=\exp(-x)\left(\sigma_0+\sigma_1x+\sigma_2x^2\right).$$

$$(11)\ E_{ch}\ :\ k\left(k^2-4\right)^2\ =\ 0\ ;\ \ k_1\ =\ 0\,,\ k_{2,3}\ =\ \pm\ 2\ ;$$

$$(11)\ S_{\rm comp}(x)=\sigma_0+\exp(2\,x)\,(\sigma_1+\sigma_2x)+\exp(-2\,x)\,(\sigma_3+\sigma_4x)\,.$$

$$(12)\ E_{ch}\ :\ k^2\left(k^2-4\right)\ =\ 0\ ;\ \ k_{1,2}\ =\ 0\,,\ k_{3,4}\ =\ \pm\ 2\ ;$$

$$(12)\ S_{\rm comp}(x)=\sigma_0\,(1+\sigma_1\,x)+\sigma_2\,\exp(2\,x)+\sigma_3\,\exp(-2\,x)\ . \tag{3.56}$$

3.3 Method of Undetermined Coefficients

Treatment of Inhomogeneous Linear Ordinary Differential Equation with Constant Coefficients

Consider

$$\sum_{j=0}^{\nu} c_j \; D^j u(x) = B(x) \;\; .$$

If $B(x)$ is nonzero, then this equation is an inhomogeneous linear ordinary differential equation with known constant coefficients, c_j.

Assume that through some fantastic luck, one has been able to guess the particular integral, I_{pi}, that solves the above equation. That is

$$\sum_{j=0}^{\nu} c_j \; D^j I_{pi} = B(x) \;\; . \tag{3.57}$$

Clearly, by using the procedure outlined in the foregoing subsections, one can also work out a complementary solution for the homogeneous part of the above inhomogeneous linear ordinary differential equation with constant coefficients. In other words, one can determine $S_{\text{comp}}(x)$ that satisfies the following equation.

$$\sum_{j=0}^{\nu} c_j \; D^j S_{\text{comp}}(x) = 0 \;\; . \tag{3.58}$$

The sum of the complementary solution $S_{\text{comp}}(x)$ and the particular integral I_{pi} leads to complete solution. That is, from (3.57) and (3.58), one has

$$\begin{aligned} \sum_{j=0}^{\nu} c_j \; D^j u(x) &= B(x) \\ = \sum_{j=0}^{\nu} c_j \; D^j S_{cs}(x) &= \sum_{j=0}^{\nu} c_j \; D^i \left[S_{\text{comp}}(x) + I_{pi} \right] \\ = \sum_{j=0}^{\nu} c_j \; D^j S_{\text{comp}}(x) + \sum_{j=0}^{\nu} c_j \; D^j I_{pi} &= 0 + B(x) = B(x) \;\; . \end{aligned} \tag{3.59}$$

3.3.1 Particular Integral: I_{pi}

While the process of determining complementary solution for homogeneous linear ordinary differential equation with constant coefficients has now been amply studied, calculation of the particular integral, I_{pi}, for inhomogeneous linear ordinary differential equations with constant coefficients still needs to be described. This matter is dealt with in the current subsection. The procedure described here is a generalization of what is known as the method of undetermined coefficients. Detailed explanation is provided by working through several calculations that involve different versions of the function $B(x)$.

Calculation of I_{pi}

$$\left[\sum_{i=0}^{\nu} c_i\ D^i\right] I_{pi}\ =\ B(x)$$

In order to solve and find the particular integral for an inhomogeneous linear ordinary differential equation with constant coefficients given above, proceed as follows: Multiply both sides on the left by $\left[\sum_{i=0}^{\nu} c_i\ D^i\right]^{-1}$. The result is

$$\left[\sum_{i=0}^{\nu} c_i\ D^i\right]^{-1}\left[\sum_{i=0}^{\nu} c_i\ D^i\right] I_{pi}\ =\ I_{pi}\ =\ \left[\sum_{i=0}^{\nu} c_i\ D^i\right]^{-1} B(x)\ . \qquad (3.60)$$

Expand $\left[\sum_{i=0}^{\nu} c_i\ D^i\right]^{-1}$ B(x) as series in ascending powers of D and retain only the terms needed for the given B(x).

3.3.2 Examples Group V

$\boldsymbol{I_{pi}}$ **when** $B(x) = e_n\, x^n + e_0$

V(A) : Solve

$$\left[D^2 + D + 1\right] u(x)\ =\ e_5\, x^5 + e_0\ .$$

Calculate first the particular integral $u(x) = I_{pi}$. To this end, as explained above, invert the left-hand side and expand the resultant in ascending powers of D, retaining terms only up to the order D^5. Note that higher powers of D in the expansion would contribute nothing.

$$V(A) \;:\; I_{pi} = \left[\frac{1}{D^2+D+1}\right](e_5\,x^5 + e_0)$$
$$= \left[1 - D + D^3 - D^4 + O(D^6)\right](e_5\,x^5 + e_0)$$
$$= e_5(x^5 - 5x^4 + 60x^2 - 120x) + e_0 \;. \tag{3.61}$$

Now calculate its complementary solution.

To that end, use the procedure previously described following (3.21) and (3.22). That is, begin by trying $\exp(k\,x)$ as a solution.

$$\left(D^2 + D + 1\right)\exp(k\,x) = (k^2 + k + 1)\exp(k\,x) \;=\; 0\,. \tag{3.62}$$

In the above equation, ignore $\exp(k\,x)$. The rest is the characteristic equation E_{ch}.

$$(k^2 + k + 1) = 0 \;. \tag{3.63}$$

The E_{ch} here has complex roots: $k = k_1 = r + i\,m$ and $k = k_2 = r - i\,m$, where

$$r = -\frac{1}{2} \quad ; \quad m = \frac{\sqrt{3}}{2} \;. \tag{3.64}$$

The resultant complementary solution consists of $\exp(k_1\,x)$ and $\exp(k_2\,x)$ and is expressed in terms of

$$\exp(r\,x)\sin(m\,x) \quad \textit{and} \quad \exp(r\,x)\cos(m\,x) \;. \tag{3.65}$$

More precisely, it is

$$V(A) \;:\; S_{\text{comp}}(x) \;=\; \exp\left(-\frac{x}{2}\right)\left[\sigma_3 \sin\left(\frac{\sqrt{3}}{2}x\right) + \sigma_4 \cos\left(\frac{\sqrt{3}}{2}x\right)\right] . \tag{3.66}$$

As usual, σ_3 and σ_4 are arbitrary constants. Also, the sign and the cosine functions are linearly independent. As such (3.61) and (3.66) represent complete solution, $= I_{pi} + S_{\text{comp}}(x)$, of differential equation V(A).

V(B): Solve

$$\left[D^2 - D - 1\right]u(x) \;=\; (e_3\,x^3 + e_0)\,.$$

The particular integral, as before, is calculated as follows:

$$V(B) \;:\qquad I_{pi} = \left[\frac{1}{D^2 - D - 1}\right](e_3\,x^3 + e_0)$$
$$= -\left[1 - D + 2D^2 - 3D^3 + O(D^4)\right](e_3\,x^3 + e_0)$$

$$= e_3\,(-x^3 + 3x^2 - 12x + 18) - e_0\ . \tag{3.67}$$

Next proceed as in exercise V(A) and find the characteristic equation.

$$(k^2 - k - 1) = 0\ . \tag{3.68}$$

Its roots, $k_{1,2} = \frac{1}{2} \pm \frac{\sqrt{5}}{2}$, lead to the complementary solution

$$V(B)\ :\ \ S_{\text{comp}}(x) = \exp\left(\frac{x}{2}\right)\left[\sigma_1 \exp\left(\frac{\sqrt{5}}{2}x\right) + \sigma_2 \exp\left(-\frac{\sqrt{5}}{2}x\right)\right]\ . \tag{3.69}$$

V(C): Solve

$$\left[D^2 + 2D + 1\right] u(x)\ =\ (e_4\,x^4 + e_0)\ .$$

The particular integral is calculated the same way as in (3.61) and (3.67).

$$\begin{aligned} V(C)\ :\qquad I_{pi} &= \left[\frac{1}{D^2 + 2D + 1}\right](e_4\,x^4 + e_0) \\ &= \left[1 - 2D + 3D^2 - 4D^3 + 5D^4 + O(D^5)\right](e_4\,x^4 + e_0) \\ &= e_4\,(x^4 - 8x^3 + 36x^2 - 96x + 120) + e_0\ . \end{aligned} \tag{3.70}$$

Next, the $S_{\text{comp}}(x)$. The characteristic equation $k^2 + 2k + 1\ =\ 0$ has a double root $k = k_1 = k_2 = -1$. Therefore, similar to (3.28)–(3.37), its complementary solution is

$$V(C)\ :\qquad S_{\text{comp}}(x) = (\sigma_0 + \sigma_1 x)\exp(-x)\ . \tag{3.71}$$

A more detailed set of equations is solved in the following.

3.3.3 Examples Group VI

Solve the following twelve inhomogeneous linear ordinary differential equations with constant coefficients. The homogeneous linear ordinary differential equations with constant coefficients that occur in (3.55) have been converted here by adding a term of the form $B(x) = e_n\,x^n + e_o$. Because $S_{\text{comp}}(x)$ to these equations have already been worked out—see (3.56)—only the particular integral, I_{pi}, needs to be worked out here. Complete solution can then be obtained by adding I_{pi} to the corresponding $S_{\text{comp}}(x)$.

$$\left(D^2 + 3D + 1\right) u(x) = (e_1\,x^3 + e_0)\ .\ (1)$$

$$\begin{aligned}
\left(D^2+3D-1\right)u(x) &= \left(e_2\,x^3+e_0\right)\ .\ (2)\\
\left(D^2+3D-3\right)u(x) &= \left(e_3\,x^3+e_0\right)\ .\ (3)\\
\left(D^2+3D+\frac{9}{2}\right)u(x) &= \left(e_4\,x^4+e_0\right)\ .\ (4)\\
\left(D^2+4D+6\right)u(x) &= \left(e_5\,x^4+e_0\right)\ .\ (5)\\
\left(D^2+D+\frac{1}{2}\right)u(x) &= \left(e_6\,x^4+e_0\right)\ .\ (6)\\
\left(D+\frac{3}{2}\right)^2 u(x) &= \left(e_7\,x^5+e_0\right)\ .\ (7)\\
\left(D-\frac{3}{2}\right)^2 u(x) &= \left(e_8\,x^5+e_0\right)\ .\ (8)\\
(D+2)^2\,u(x) &= \left(e_9\,x^5+e_0\right)\ .\ (9)\\
(D+1)^3\,u(x) &= \left(e_{10}\,x^6+e_0\right)\ .\ (10)\\
\left[D\,(D^2-4)^2\right]u(x) &= \left(e_{11}\,x^6+e_0\right)\ .\ (11)\\
\left[D^2\,(D^2-1)\right]u(x) &= \left(e_{12}\,x^6+e_0\right)\ .\ (12)
\end{aligned} \tag{3.72}$$

Solution: (3.72)

$$\begin{aligned}
(1)\ I_{pi} &= \left(D^2+3D+1\right)^{-1}\left(e_1\,x^3+e_0\right)\\
&= \left[1-3D+8D^2-21D^3+\ldots\right]\left(e_1\,x^3+e_0\right)\ ,\\
&= e_1(x^3-9x^2+48x-126)+e_0\ .\\
(2)\ I_{pi} &= \left(D^2+3D-1\right)^{-1}\left(e_2\,x^3+e_0\right)\\
&= \left[-1-3D-10D^2-33D^3+\ldots\right]\left(e_2\,x^3+e_0\right)\ ,\\
&= -e_2(x^3+9x^2+60x+198)-e_0\ .\\
(3)\ I_{pi} &= \left(D^2+3D-3\right)^{-1}\left(e_3\,x^3+e_0\right)\\
&= \left(\frac{1}{3}\right)\left[-1-D-\frac{4}{3}D^2-\frac{5}{3}D^3+\ldots\right]\left(e_3\,x^3+e_0\right)\ ,\\
&= -\left(\frac{e_3}{3}\right)\left(x^3+3x^2+8x+10\right)-\frac{e_0}{3}\ .\\
(4)\ I_{pi} &= \left(D^2+3D+\frac{9}{2}\right)^{-1}\left(e_4\,x^4+e_0\right)\\
&= \left(\frac{2}{9}\right)\left[1-\frac{2}{3}D+\frac{2}{9}D^2-\frac{4}{81}D^4+\ldots\right]\left(e_4\,x^4+e_0\right)\ ,\\
&= e_4\left(\frac{2}{9}\right)\left(x^4-\frac{8}{3}x^3+\frac{8}{3}x^2-\frac{32}{27}\right)+\left(\frac{2}{9}\right)e_0\ .\\
(5)\ I_{pi} &= \left(D^2+4D+6\right)^{-1}\left(e_5\,x^4+e_0\right)
\end{aligned}$$

$$= \left(\frac{1}{6}\right)\left[1 - \frac{2}{3}D + \frac{5}{18}D^2 - \frac{2}{27}D^3 + \frac{1}{324}D^4 + ...\right](e_5\,x^4 + e_0) \ ,$$

$$= \frac{e_5}{6}\left(x^4 - \frac{8}{3}x^3 + \frac{10}{3}x^2 - \frac{16}{9}x + \frac{2}{27}\right) + \frac{e_0}{6} \ .$$

(6) $$I_{pi} = \left(D^2 + D + \frac{1}{2}\right)^{-1}(e_6\,x^4 + e_0)$$

$$= 2\,e_6\left[1 - 2D + 2D^2 - 4D^4 + ...\right]x^4 + 2\,e_0 \ ,$$

$$= 2\,e_6\left(x^4 - 8x^3 + 24x^2 - 96\right) + 2\,e_0 \ .$$

(7) $$I_{pi} = \left(D + \frac{3}{2}\right)^{-2}(e_7\,x^5 + e_0)$$

$$= \frac{4}{9}\left[1 - \frac{4}{3}D + \frac{4}{3}D^2 - \frac{32}{27}D^3 + \frac{80}{81}D^4 - \frac{64}{81}D^5 + ...\right](e_7\,x^5 + e_0) \ ,$$

$$= \frac{4}{9}\,e_7\left(x^5 - \frac{20}{3}x^4 + \frac{80}{3}x^3 - \frac{640}{9}x^2 + \frac{3200}{27}x - \frac{2560}{27}\right) + \frac{4}{9}\,e_0 \ .$$

(8) $$I_{pi} = \left(D - \frac{3}{2}\right)^{-2}(e_8\,x^5 + e_0)$$

$$= \frac{4}{9}\left[1 + \frac{4}{3}D + \frac{4}{3}D^2 + \frac{32}{27}D^3 + \frac{80}{81}D^4 + \frac{64}{81}D^5 + ...\right](e_8\,x^5 + e_0) \ ,$$

$$= \frac{4}{9}\,e_8\left(x^5 + \frac{20}{3}x^4 + \frac{80}{3}x^3 + \frac{640}{9}x^2 + \frac{3200}{27}x + \frac{2560}{27}\right) + \frac{4}{9}\,e_0 \ .$$

(9) $$I_{pi} = (D+2)^{-2}(e_9\,x^5 + e_0)$$

$$= \frac{1}{4}\left[1 - D + \frac{3}{4}D^2 - \frac{1}{2}D^3 + \frac{5}{16}D^4 - \frac{3}{16}D^5 + ...\right](e_9\,x^5 + e_0) \ ,$$

$$= \frac{1}{4}\,e_9\left(x^5 - 5x^4 + 15x^3 - 30x^2 + \frac{75}{2}x - \frac{45}{2}\right) + \frac{1}{4}\,e_0 \ .$$

(10) $$I_{pi} = (D+1)^{-3}(e_{10}\,x^6 + e_0)$$

$$= \left[1 - 3D + 6D^2 - 10D^3 + 15D^4 - 21D^5 + 28D^6 - ...\right](e_{10}\,x^6 + e_0) \ ,$$

$$= e_{10}\,(x^6 - 18x^5 + 180x^4 - 1200x^3 + 5400x^2 - 15120x + 20160) + e_0 \ .$$

(11) $$I_{pi} = \left[D\,(D^2-4)^2\right]^{-1}(e_{11}\,x^6 + e_0)$$

$$= \left[\frac{1}{D} + 8\,D + 63\,D^3 + 496\,D^5 + O(D^7)\right](e_{11}\,x^6 + e_0)$$

$$= \left\{e_0\,x + e_{11}\left[\frac{x^7}{7} + 48\,x^5 + 7560\,x^3 + 357120\,x\right]\right\} \ . \qquad (11)$$

(12) $$I_{pi} = \left[D^2\,(D^2-1)\right]^{-1}(e_{12}\,x^6 + e_0)$$

$$= -\left[\frac{1}{D^2} + 1 + D^2 + D^4 + D^6 + O(D^8)\right](e_{12}\,x^6 + e_0)$$

$$= -\left\{e_0\left(\frac{x^2}{2}+1\right)+e_{12}\left[\frac{x^8}{56}+x^6+30x^4+360x^2+720\right]\right\}. \quad (12)\ (3.73)$$

Note, in (11) above use was made of the relationship

$$\frac{1}{D}f(x) \;=\; \int f(x)\,\mathrm{d}x \;,$$

and in (12)

$$\frac{1}{D^2}f(x) \;=\; \int \mathrm{d}x\left\{\int f(x)\,\mathrm{d}x\right\}.$$

Therefore

$$\frac{1}{D}(e_{11}\,x^6+e_0) \;=\; \frac{e_{11}}{7}x^7+e_0x$$

and

$$\frac{1}{D^2}(e_{12}\,x^6+e_0) \;=\; \frac{e_{12}}{56}x^8+\frac{e_0}{2}x^2\,.$$

In keeping with the tradition, I_{pi} contains only known constants e_i, etc. Unknown constants such as σ_i, etc., are not included in the result above. That is appropriate because such constants are already equivalently present in the relevant $S_{\text{comp}}(x)$ in (3.56) and therefore in the complete solution.

3.3.4 Problems Group II

Work out the particular integral I_{pi} for the following ten inhomogeneous linear ordinary differential equations with constant coefficients. The equations given below were put together by making additions—in the form $B(x)=c_n\,x^n+c_0$—to the homogeneous linear ordinary differential equation with constant coefficients given in Problems Group I. Because the relevant complementary solutions are already available from Problems Group I, only particular integrals, I_{pi}, are being required here.

$$\left(D^2+2D-3\right)u(x)=c_1\,x^3+c_0\;.\;(1)$$
$$\left(D^2-3D-4\right)u(x)=c_2\,x^3+c_0\;.\;(2)$$
$$\left(D+\frac{1}{2}\right)^2u(x)=c_3\,x^3+c_0\;.\;(3)$$
$$\left(D-\frac{1}{2}\right)^2u(x)=c_4\,x^4+c_0\;.\;(4)$$

$$(D+2)^3\,u(x) = c_5\,x^4 + c_0\ . \quad (5)$$
$$(D-2)^3\,u(x) = c_6\,x^4 + c_0\ . \quad (6)$$
$$\left(D^2+D+1\right)u(x) = c_7\,x^5 + c_0\ . \quad (7)$$
$$\left(D^2-D+1\right)u(x) = c_8\,x^5 + c_0\ . \quad (8)$$
$$\left(D^2+2D+3\right)u(x) = c_9\,x^5 + c_0\ . \quad (9)$$
$$\left(D^2+3D+4\right)u(x) = c_{10}\,x^6 + c_0\ . \quad (10)$$

Procedure for Solving

I_{pi} **When** $B(x) = e_n\ \exp(\alpha x) + e_0$

In order to solve an inhomogeneous linear ordinary differential equation with constant coefficients where $B(x) = e_n\ \exp(\alpha\,x) + e_0$, it is helpful to know the effect of operating with a function of D on $e_n\ \exp(\alpha\,x)$. To that purpose, consider the following identity for integral $n \geq 0$.

$$D^n\left[e_n\ \exp(\alpha\,x)\right] = e_n\,\alpha^n \exp(\alpha\,x)\ . \quad (3.74)$$

As such, when acting on $\exp(\alpha\,x)$ a function $f(D)$ that involves only positive powers of D will lead to

$$f(D)\left[e_n\ \exp(\alpha\,x)\right] = e_n\ f(\alpha)\ \exp(\alpha\,x)\ .$$

If $f(\alpha) \neq 0$, one can divide both sides by $f(\alpha)$.

$$\frac{1}{f(\alpha)}f(D)\ \left[e_n\ \exp(\alpha\,x)\right] = \frac{e_n\ f(\alpha)\exp(\alpha\,x)}{f(\alpha)}\ =\ e_n\ \exp(\alpha\,x). \quad (3.75)$$

From the left, multiply both sides of (3.75) by $\frac{1}{f(D)}$.

$$\frac{1}{f(D)}\frac{1}{f(\alpha)}f(D)\ \left[e_n\ \exp(\alpha\,x)\right] = \frac{1}{f(D)}\left[e_n\exp(\alpha\,x)\right]. \quad (3.76)$$

And rewrite the left-hand side of (3.76) in an equivalent form.

$$\frac{1}{f(\alpha)}\left\{\frac{1}{f(D)}f(D)\ \left[e_n\ \exp(\alpha\,x)\right]\right\} = \frac{e_n\ \exp(\alpha\,x)}{f(\alpha)}\ . \quad (3.77)$$

The left-hand sides of (3.76) and (3.77) are equal and hence, also the right-hand sides.

$$\frac{1}{f(D)}\left[e_n\exp(\alpha\,x)\right] = \frac{e_n\ \exp(\alpha\,x)}{f(\alpha)}\ . \quad (3.78)$$

3.3.5 *Examples Group VII*

Here $B(x) = e_n \exp(\alpha x) + e_0$

Solve the following twelve inhomogeneous linear ordinary differential equations with constant coefficients. For convenience, the homogeneous part of these equations is the same as (3.55). For calculating the relevant I_{pi}, it is helpful to use (3.61), (3.67), (3.70) and (3.78).

$$\begin{aligned}
\left(D^2+3D+1\right)u(x) &= e_1 \exp(\alpha x)+e_0 \ . && (1)\\
\left(D^2+3D-1\right)u(x) &= e_2 \exp(\alpha x)+e_0 \ . && (2)\\
\left(D^2+3D-3\right)u(x) &= e_3 \exp(\alpha x)+e_0 \ . && (3)\\
\left(D^2+3D+\frac{9}{2}\right)u(x) &= e_4 \exp(\alpha x)+e_0 \ . && (4)\\
\left(D^2+4D+6\right)u &= e_5 \exp(\alpha x)+e_0 \ . && (5)\\
\left(D^2+D+\frac{1}{2}\right)u(x) &= e_6 \exp(\alpha x)+e_0 \ . && (6)\\
\left(D+\frac{3}{2}\right)^2 u(x) &= e_7 \exp(\alpha x)+e_0 \ . && (7)\\
\left(D-\frac{3}{2}\right)^2 u(x) &= e_8 \exp(\alpha x)+e_0 \ . && (8)\\
(D+2)^2\,u(x) &= e_9 \exp(\alpha x)+e_0 \ . && (9)\\
(D+1)^3\,u(x) &= e_{10} \exp(\alpha x)+e_0 \ . && (10)\\
\left[D\,(D^2-4)^2\right]u(x) &= e_{11} \exp(\alpha x)+e_0 \ . && (11)\\
\left[D^2\,(D^2-1)\right]u(x) &= e_{12} \exp(\alpha x)+e_0 \ . && (12)
\end{aligned} \qquad (3.79)$$

Solution: I_{pi} **for** (3.79)

$$\begin{aligned}
I_{pi} &= \frac{1}{\left(D^2+3D+1\right)}\left(e_1 \exp(\alpha x)+e_0\right) = \frac{e_1 \exp(\alpha x)}{\left(\alpha^2+3\alpha+1\right)}+\frac{e_0}{1} \ . && (1)\\
I_{pi} &= \frac{1}{\left(D^2+3D-1\right)}\left(e_2 \exp(\alpha x)+e_0\right) = \frac{e_2 \exp(\alpha x)}{\left(\alpha^2+3\alpha-1\right)}+\frac{e_0}{-1} \ . && (2)\\
I_{pi} &= \frac{1}{\left(D^2+3D-3\right)}\left(e_3 \exp(\alpha x)+e_0\right) = \frac{e_3 \exp(\alpha x)}{\left(\alpha^2+3\alpha-3\right)}+\frac{e_0}{-3} \ . && (3)\\
I_{pi} &= \frac{1}{\left(D^2+3D+\frac{9}{2}\right)}\left(e_4 \exp(\alpha x)+e_0\right) = \frac{e_4 \exp(\alpha x)}{\left(\alpha^2+3\alpha+\frac{9}{2}\right)}+\frac{e_0}{9/2} \ . && (4)\\
I_{pi} &= \frac{1}{\left(D^2+4D+6\right)}\left(e_5 \exp(\alpha x)+e_0\right) = \frac{e_5 \exp(\alpha x)}{\left(\alpha^2+4\alpha+6\right)}+\frac{e_0}{6} \ . && (5)
\end{aligned}$$

$$I_{pi} = \frac{1}{\left(D^2+D+\frac{1}{2}\right)}\left(e_6\exp(\alpha x)+e_0\right) = \frac{e_6\,\exp(\alpha x)}{\left(\alpha^2+\alpha+\frac{1}{2}\right)} + \frac{e_0}{1/2} \quad . \qquad (6)$$

$$I_{pi} = \frac{1}{\left(D+\frac{3}{2}\right)^2}\left(e_7\,\exp(\alpha x)+e_0\right) = \frac{e_7\,\exp(\alpha x)}{\left(\alpha+\frac{3}{2}\right)^2} + \frac{e_0}{9/4} \quad . \qquad (7)$$

$$I_{pi} = \frac{1}{\left(D-\frac{3}{2}\right)^2}\left(e_8\,\exp(\alpha x)+e_0\right) = \frac{e_8\,\exp(\alpha x)}{\left(\alpha-\frac{3}{2}\right)^2} + \frac{e_0}{9/4} \quad . \qquad (8)$$

$$I_{pi} = \frac{1}{(D+2)^2}\left(e_9\,\exp(\alpha x)+e_0\right) = \frac{e_9\,\exp(\alpha x)}{(\alpha+2)^2} + \frac{e_0}{4} \quad . \qquad (9)$$

$$I_{pi} = \frac{1}{(D+1)^3}\left(e_{10}\,\exp(\alpha x)+e_0\right) = \frac{e_{10}\,\exp(\alpha x)}{(\alpha+1)^3} + \frac{e_0}{1} \quad . \qquad (10)$$

$$I_{pi} = \frac{1}{D\left(D^2-4\right)^2}(e_{11}\,\exp(\alpha x)+e_0) = \frac{1}{D\left(D^2-4\right)^2}[e_{11}\,\exp(\alpha x)]$$
$$+\frac{1}{16\,D}\left[1-\frac{D^2}{2}\right]^{-1} e_0 = \frac{e_{11}\,\exp(\alpha x)}{\alpha\left(\alpha^2-4\right)^2} + e_0\left(\frac{x}{16}\right) \quad . \qquad (11)$$

$$I_{pi} = \frac{1}{D^2\left(D^2-1\right)}\left(e_{12}\,\exp(\alpha x)+e_0\right) = \frac{1}{D^2\left(D^2-1\right)}[e_{12}\,\exp(\alpha x)]$$
$$+\frac{1}{-D^2}\left[1-D^2\right]^{-1} e_0 = \frac{e_{12}\,\exp(\alpha x)}{\alpha^2\left(\alpha^2-1\right)} - e_0\left(\frac{x^2}{2}\right) - e_0 \quad . \qquad (12) \qquad (3.80)$$

3.3.6 Problems Group III

The inhomogeneous linear ordinary differential equations with constant coefficients given below are obtained by making additions—in the form $B(x) = c_n\,\exp(\alpha x) + c_0$—to the homogeneous linear ordinary differential equations with constant coefficients given in Problems Group I. Because the relevant complementary solutions are already available from Problems Group I, only particular integrals, I_{pi}, are needed here.

$$\left(D^2+2D-3\right)u(x) = c_1\,\exp(\alpha x)+c_0 \;.\; (1)$$
$$\left(D^2-3D-4\right)u(x) = c_2\,\exp(\alpha x)+c_0 \;.\; (2)$$
$$\left(D+\frac{1}{2}\right)^2 u(x) = c_3\,\exp(\alpha x)+c_0 \;.\; (3)$$
$$\left(D-\frac{1}{2}\right)^2 u(x) = c_4\,\exp(\alpha x)+c_0 \;.\; (4)$$
$$(D+2)^3\,u(x) = c_5\,\exp(\alpha x)+c_0 \;.\; (5)$$
$$(D-2)^3\,u(x) = c_6\,\exp(\alpha x)+c_0 \;.\; (6)$$

$$\left(D^2 + D + 1\right) u(x) = c_7 \exp(\alpha x) + c_0 \ . \ (7)$$
$$\left(D^2 - D + 1\right) u(x) = c_8 \exp(\alpha x) + c_0 \ . \ (8)$$
$$\left(D^2 + 2D + 3\right) u(x) = c_9 \exp(\alpha x) + c_0 \ . \ (9)$$
$$\left(D^2 + 3D + 4\right) u(x) = c_{10} \exp(\alpha x) + c_0 \ . \ (10)$$

3.3.7 I_{pi} for $B(x) = \cos(\alpha x)\,;\ \sin(\alpha x)$

Procedure for Solving

In order to solve and find the particular integral for a given inhomogeneous linear ordinary differential equation with $B(x) = \cos(\alpha x)$, $\sin(\alpha x)$, etc., use the following relationships.

$$\cos(\alpha x) = \frac{1}{2}\left[\exp(i\alpha x) + \exp(-i\alpha x)\right] \ ,$$
$$\sin(\alpha x) = \frac{1}{2i}\left[\exp(i\alpha x) - \exp(-i\alpha x)\right] \ . \qquad (3.81)$$

This renders the inhomogeneous terms similar to those treated in detail in (3.79) and (3.80).

3.3.8 Examples Group VIII

Use (3.81) and information provided in (3.78) to solve for the inhomogeneous linear ordinary differential equations with constant coefficients given below.

$$\left(D^2 + 3D + 1\right) u(x) = 2\cos(x) \ . \ (1)$$
$$\left(D^2 + 3D - 1\right) u(x) = 2i \sin(x) \ . \ (2)$$
$$\left(D^2 + 3D - 3\right) u(x) = 2\cos(x) \ . \ (3)$$
$$\left(D^2 + 3D + \frac{9}{2}\right) u(x) = 2\cos(x) \ . \ (4)$$
$$\left(D^2 + 4D + 6\right) u(x) = 2i \sin(x) \ . \ (5)$$
$$\left(D^2 + D + \frac{1}{2}\right) u(x) = 2i \sin(x) \ . \ (6)$$
$$\left(D + \frac{3}{2}\right)^2 u(x) = 4\cos^2(x) \ . \ (7)$$
$$\left(D - \frac{3}{2}\right)^2 u(x) = 4\sin^2(x) \ . \ (8)$$

$$
\begin{aligned}
(D+2)^2\,u(x) &= 2\cos(2x)\ . \quad (9)\\
(D+1)^3\,u(x) &= 2i\,\sin(2x)\ . \quad (10)\\
\left[D\,(D^2-4)\right]u(x) &= 4\cos^2(2x)\ . \quad (11)\\
\left[D^2\,(D^2-1)\right]u(x) &= -4\sin^2(2x)\ . \quad (12)
\end{aligned}
\tag{3.82}
$$

I_{pi} for (3.82)

Because the $S_{\text{comp}}(x)$ have already been calculated for the differential equations (3.82)—see (3.55) and (3.56)—only the I_{pi} are worked out here.

$$
\begin{aligned}
(1)\ I_{pi}:\ \frac{2}{\left(D^2+3D+1\right)}\cos(x) &= \frac{1}{\left(D^2+3D+1\right)}\left[\exp(ix)+\exp(-ix)\right]\\
\frac{\exp(ix)}{3i}-\frac{\exp(-ix)}{3i} &= \frac{2}{3}\sin(x)\ .
\end{aligned}
$$

$$
\begin{aligned}
(2)\ I_{pi}:\ \frac{1}{\left(D^2+3D-1\right)}2i\,\sin(x) &= \frac{1}{\left(D^2+3D-1\right)}\left[\exp(ix)-\exp(-ix)\right]\\
= -\left[\frac{\exp(ix)}{2-3i}-\frac{\exp(-ix)}{2+3i}\right] &= \left(\frac{-i}{13}\right)\left[4\sin(x)+6\cos(x)\right]\ .
\end{aligned}
$$

$$
\begin{aligned}
(3)\ I_{pi}:\ \frac{1}{\left(D^2+3D-3\right)}2\cos(x) &= \frac{1}{\left(D^2+3D-3\right)}\left[\exp(ix)+\exp(-ix)\right]\\
= -\left[\frac{\exp(ix)}{4-3i}-\frac{\exp(-ix)}{4+3i}\right] &= \left(\frac{2}{25}\right)\left[-4\cos(x)+3\sin(x)\right]\ .
\end{aligned}
$$

$$
\begin{aligned}
(4)\ I_{pi}:\ \frac{1}{\left(D^2+3D+\frac{9}{2}\right)}2\cos(x) &= \frac{1}{\left(D^2+3D+\frac{9}{2}\right)}\left[\exp(ix)+\exp(-ix)\right]\\
= \left[\frac{\exp(ix)}{\frac{7}{2}+3i}+\frac{\exp(-ix)}{\frac{7}{2}-3i}\right] &= \left(\frac{4}{85}\right)\left[6\sin(x)+7\cos(x)\right]\ .
\end{aligned}
$$

$$
\begin{aligned}
(5)\ I_{pi}:\ \frac{2i}{\left(D^2+4D+6\right)}\sin(x) &= \frac{1}{\left(D^2+4D+6\right)}\left[\exp(ix)-\exp(-ix)\right]\\
= \left[\frac{\exp(ix)}{5+4i}-\frac{\exp(-ix)}{5-4i}\right] &= \left(\frac{i}{41}\right)\left[10\sin(x)-8\cos(x)\right]\ .
\end{aligned}
$$

$$
\begin{aligned}
(6)\ I_{pi}:\ \frac{1}{\left(D^2+D+\frac{1}{2}\right)}\left[2i\,\sin(x)\right] &= \frac{1}{\left(D^2+D+\frac{1}{2}\right)}\left[\exp(ix)-\exp(-ix)\right]\\
= \left[\frac{\exp(ix)}{\frac{3}{2}+i}-\frac{\exp(-ix)}{\frac{3}{2}-i}\right] &= -\left(\frac{4i}{5}\right)\left[\sin(x)+2\cos(x)\right]\ .
\end{aligned}
$$

$$(7)\ I_{pi}:\ \frac{1}{\left(D+\frac{3}{2}\right)^2}[4\cos^2(x)] = \frac{1}{\left(D+\frac{3}{2}\right)^2}[\exp(2ix)+\exp(-2ix)+2]$$
$$= \frac{\exp(2ix)}{\left(2i+\frac{3}{2}\right)^2} + \frac{\exp(-2ix)}{\left(-2i+\frac{3}{2}\right)^2} + \frac{2}{\left(\frac{3}{2}\right)^2} = \frac{192}{625}\sin(2x) - \frac{56}{625}\cos(2x) + \frac{8}{9}\ .$$

$$(8)\ I_{pi}:\ \frac{1}{\left(D-\frac{3}{2}\right)^2}[4\sin^2(x)] = \frac{1}{\left(D-\frac{3}{2}\right)^2}[2-\exp(2ix)-\exp(-2ix)]$$
$$= \frac{2}{\left(\frac{3}{2}\right)^2} + \frac{\exp(2ix)}{\left(2i+\frac{3}{2}\right)^2} + \frac{\exp(-2ix)}{\left(-2i+\frac{3}{2}\right)^2} = \frac{8}{9} + \frac{192}{625}\sin(2x) + \frac{56}{625}\cos(2x)\ .$$

$$(9)\ I_{pi}:\ \frac{1}{(D+2)^2}[2\cos(2x)] = \frac{1}{4}\left[\frac{\exp(2ix)}{(1+i)^2} + \frac{\exp(-2ix)}{(1-i)^2}\right]$$
$$= \frac{\sin(2x)}{4}.$$

$$(10)\ I_{pi}:\ \frac{1}{(D+1)^3}[2i\sin(2x)] = \frac{\exp(2ix)-\exp(-2ix)}{(D+1)^3}$$
$$= \frac{\exp(2ix)}{(1+2i)^3} - \frac{\exp(-2ix)}{(1-2i)^3} = -\left(\frac{2i}{125}\right)[11\sin(2x) - 2\cos(2x)]\,.$$

$$(11)\ I_{pi}:\ \frac{1}{D\left(D^2-4\right)}[4\cos^2(2x)] = \frac{1}{D(D^2-4)}\left[\exp(4ix)+\exp(-4ix)\right]$$
$$+\left[\frac{1}{D^3-4D}\right]2 = \frac{1}{4i}\left[\frac{\exp(4ix)}{(4i)^2-4}\right]$$
$$-\frac{1}{4i}\left[\frac{\exp(-4ix)}{(-4i)^2-4}\right] + \frac{1}{-4D}\left[1-\frac{D^2}{4}\right]^{-1}2$$
$$= -\frac{\sin(4x)}{40} - \frac{x}{2}\ .$$

$$(12)\ I_{pi}:\ \frac{-4\sin^2(2x)}{D^2\left(D^2-1\right)} = \frac{1}{D^2}\left[\frac{\exp(4ix)}{D^2-1} + \frac{\exp(-4ix)}{D^2-1}\right]$$
$$-\left[\frac{1}{D^4-D^2}\right]2 = \frac{1}{(4i)^2}\left[\frac{\exp(4ix)}{(4i)^2-1}\right]$$
$$+\frac{1}{(4i)^2}\left[\frac{\exp(-4ix)}{(-4i)^2-1}\right] + \left[\frac{1}{D^2}\right]2 - 2 = \frac{\cos(4x)}{136} + x^2 - 2\ . \tag{3.83}$$

3.3.9 Problems Group IV

Additions have been made to the right-hand side in the form $B(x) = \sin(nx)$ and $\cos(nx)$ to the homogeneous linear ordinary differential equations with constant coefficients in Problems Group I. Work out the particular integrals, I_{pi}, for the following differential equations.

$$\left(D^2 + 2D - 3\right) u(x) = 2\,\cos(x)\ . \quad (1)$$
$$\left(D^2 - 3D - 4\right) u(x) = 2\,i\ \sin(x)\ . \quad (2)$$
$$\left(D + \frac{1}{2}\right)^2 u(x) = \sin(x)\ \sin(2\,x)\ . \quad (3)$$
$$\left(D - \frac{1}{2}\right)^2 u(x) = \sin(x)\ \cos(2x)\ . \quad (4)$$
$$(D+2)^3\, u(x) = 4\cos^2(x)\ . \quad (5)$$
$$(D-2)^3\, u(x) = -4\sin^2(x)\ . \quad (6)$$
$$\left(D^2 + D + 1\right) u(x) = 2\cos(2x)\ . \quad (7)$$
$$\left(D^2 - D + 1\right) u(x) = 2i\,\sin(2x)\ . \quad (8)$$
$$\left(D^2 + 2D + 3\right) u(x) = 4\cos^2(2x)\ . \quad (9)$$
$$\left(D^2 + 3D + 4\right) u(x) = -4\sin^2(2x)\ . \quad (10)$$

3.3.10 I_{pi} for $B(x) = \exp(\alpha\, x)\, W(x)$

Knowing

$$\left\{D^m \exp(\alpha\, x)\right\}\ W(x) = \exp(\alpha\, x)\,(D+\alpha)^m\ W(x)\,,$$

one can write

$$\left\{[\Phi(D)]^n \exp(\alpha\, x)\right\}\ W(x) = \{\exp(\alpha\, x)[\Phi(D+\alpha)]^n\}\, W(x) \qquad (3.84)$$

where $\Phi(D)$ is a function involving positive powers of D. Therefore, one calculates the particular integral for $B(x) = \exp(\alpha\, x)\, W(x)$ by following the same strategy as employed in (3.75)–(3.78). That is, set $n = -1$.

$$\begin{aligned} I_{pi} &= \left\{[\Phi(D)]^{-1} \exp(\alpha\, x)\right\}\ W(x) = \{\exp(\alpha\, x)[\Phi(D+\alpha)]^{-1}\}\, W(x) \\ &= \left\{\frac{\exp(\alpha\, x)}{[\Phi(D)]}\right\}\ W(x) = \left\{\frac{\exp(\alpha\, x)}{[\Phi(D+\alpha)]}\right\}\ W(x) \quad . \end{aligned} \qquad (3.85)$$

As a simple exercise, consider differential equation with $B(x) = \exp(x)\, 2x^3$.

$$\left[D^2 + 2\,D + 1\right] u = \exp(x)\, 2x^3 \quad . \tag{3.86}$$

Notice $\alpha = 1$ here.

$$\begin{aligned} I_{pi} &= \left\{ \frac{1}{\left[D^2 + 2\,D + 1\right]} \exp(x) \right\} 2x^3 \\ &= \exp(x) \left\{ \frac{1}{(D+1)^2 + 2\,(D+1) + 1} \right\} 2\,x^3 \\ &= \frac{\exp(x)}{2} \left\{ \frac{1}{\left(1 + D + \frac{D^2}{4}\right)} x^3 \right\} \\ &= \frac{\exp(x)}{2} \left\{ 1 - D + \frac{3}{4} D^2 - \frac{1}{2} D^3 + O(D^4) \right\} x^3 \\ I_{pi} &= \frac{\exp(x)}{2} \left(x^3 - 3\,x^2 + \frac{9}{2}\,x - 3 \right) \quad . \end{aligned} \tag{3.87}$$

The characteristic equation, E_{ch}, is found from the differential equation in the usual manner. The complementary solution, $S_{\text{comp}}(x)$, is found according to (3.39)–(3.42). Both are given below in (3.88).

$$\begin{aligned} E_{ch} \quad : \qquad & k^2 + 2\,k + 1 = 0 \ ; \\ & \left[D^2 + 2\,D + 1\right] u = 0 \ . \end{aligned} \tag{3.88}$$

The characteristic equation has double roots $k = -1$. Therefore, just as in (3.37), the complementary solution is

$$S_{\text{comp}}(x) = \exp(-x)(\sigma_1\, x + \sigma_0) \quad . \tag{3.89}$$

The complete solution, $S_{cs}(x)$, is the sum of $S_{\text{comp}}(x)$, (3.89), and the I_{pi}, (3.87).

3.3.11 Examples Group IX

Again treat the same twelve homogeneous linear ordinary differential equations with constant coefficients as in (3.55) and change them into inhomogeneous linear ordinary differential equations with constant coefficients by adding $B(x) = \exp(\alpha\, x)\, W(x)$. The above procedure can readily be generalized to deal with cases where $B(x)$ is either $\exp(\alpha\, x)\{\cos(x)\}^{\mu}\, x^n$ or $\exp(\alpha\, x)\{\sin(x)\}^{\mu}\, x^n$. That is so because $B(x)$ can again be expressed in terms of such relationships as $\exp(\alpha\, x + \nu)\, x^n$. In this spirit, solve the following set of twelve equations (3.90).

$$\begin{aligned}
\left(D^2+3D+1\right)u(x) &= \exp(3x)\,x^2 \;. \quad (1)\\
\left(D^2+3D-1\right)u(x) &= \exp(3x)\,x^2 \;. \quad (2)\\
\left(D^2+3D-3\right)u(x) &= \exp(3x)\,x^2 \;. \quad (3)\\
\left(D^2+3D+\frac{9}{2}\right)u(x) &= \exp(4x)\,x^2 \;. \quad (4)\\
\left(D^2+4D+6\right)u(x) &= \exp(4x)\,x^2 \;. \quad (5)\\
\left(D^2+D+\frac{1}{2}\right)u(x) &= \exp(4x)\,x^2 \;. \quad (6)\\
\left(D+\frac{3}{2}\right)^2 u(x) &= 2\exp(-x)\,\cos(x)\,x \;. \quad (7)\\
\left(D-\frac{3}{2}\right)^2 u(x) &= 2\exp(x)\,\cos(x)\,x \;. \quad (8)\\
(D+2)^2\,u(x) &= 2i\exp(-x)\,\sin(x)\,x \;. \quad (9)\\
(D+1)^3\,u(x) &= 2i\exp(x)\,\sin(x)\,x \;. \quad (10)\\
\left[D\left(D^2-4\right)\right]u(x) &= 4\cos^2(2x)\,x^2 \;. \quad (11)\\
\left[D^2\left(D^2-1\right)\right]u(x) &= -4\sin^2(2x)\,x^2 \;. \quad (12)
\end{aligned} \tag{3.90}$$

I_{pi} for (3.90)

We use (3.84), (3.85), and (3.86). Again, because $S_{\text{comp}}(x)$ has already been calculated for the differential equations (3.90)—see (3.55) and (3.56)—only the I_{pi} are being worked out here.

$$\begin{aligned}
(1)\;\; I_{pi}: \quad \frac{1}{\left\{D^2+3D+1\right\}}\exp(3x)\,x^2 &= \exp(3x)\frac{1}{\left\{(D+3)^2+3(D+3)+1\right\}}\,x^2\\
= \exp(3x)\frac{1}{\left\{D^2+9D+19\right\}}\,x^2 \quad &= \left[\frac{\exp(3x)}{19}\right]\left\{1-\frac{9D}{19}+\frac{62D^2}{361}\right\}x^2\\
&= \left(\frac{\exp(3x)}{19}\right)\left\{x^2-\frac{18}{19}x+\frac{124}{361}\right\}. \quad (1)
\end{aligned}$$

$$\begin{aligned}
(2)\;\; I_{pi}: \quad \frac{1}{\left\{D^2+3D-1\right\}}\exp(3x)\,x^2 &= \exp(3x)\frac{1}{\left\{(D+3)^2+3(D+3)-1\right\}}\,x^2\\
= \exp(3x)\frac{1}{\left\{D^2+9D+17\right\}}\,x^2 \quad &= \left[\frac{\exp(3x)}{17}\right]\left\{1-\frac{9D}{17}+\frac{64D^2}{289}\right\}x^2\\
&= \left(\frac{\exp(3x)}{17}\right)\left\{x^2-\frac{18}{17}x+\frac{128}{289}\right\}. \quad (2)
\end{aligned}$$

$$(3)\;\; I_{pi}: \quad \frac{1}{\left\{D^2+3D-3\right\}}\exp(3x)\,x^2 = \exp(3x)\frac{1}{\left\{(D+3)^2+3(D+3)-3\right\}}\,x^2$$

$$= \exp(3x)\frac{1}{\left\{D^2+9D+15\right\}}x^2 \quad = \left[\frac{\exp(3x)}{15}\right]\left\{1-\frac{3D}{5}+\frac{22D^2}{75}\right\}x^2$$
$$= \left(\frac{\exp(3x)}{15}\right)\left\{x^2-\frac{6}{5}x+\frac{44}{75}\right\}.(3)$$

$$(4)\ I_{pi}: \ \frac{1}{\left\{D^2+3D+\frac{9}{2}\right\}}\exp(4x)\,x^2 = \exp(4x)\frac{1}{\left\{(D+4)^2+3(D+4)+\frac{9}{2}\right\}}x^2$$
$$= \exp(4x)\frac{1}{\left\{D^2+11D+\frac{65}{2}\right\}}x^2 = 2\left[\frac{\exp(4x)}{65}\right]\left\{1-\frac{22}{65}D+\frac{354}{(65)^2}D^2\right\}x^2$$
$$= 2\left(\frac{\exp(4x)}{65}\right)\left\{x^2-\frac{44}{65}x+\frac{708}{(65)^2}\right\}.(4)$$

$$(5)\ I_{pi}: \ \frac{1}{\left(D^2+4D+6\right)}\exp(4x)\,x^2 = \exp(4x)\frac{1}{\left\{(D+4)^2+4(D+4)+6\right\}}x^2$$
$$= \exp(4x)\frac{1}{\left\{D^2+12D+38\right\}}x^2 = \left[\frac{\exp(4x)}{38}\right]\left\{1-\frac{6}{19}D+\frac{53}{722}D^2\right\}x^2$$
$$= \left(\frac{\exp(4x)}{38}\right)\left\{x^2-\frac{12}{19}x+\frac{53}{(19)^2}\right\}.(5)$$

$$(6)\ I_{pi}: \ \frac{1}{\left(D^2+D+\frac{1}{2}\right)}\exp(4x)\,x^2 = \exp(4x)\frac{1}{\left\{(D+4)^2+(D+4)+\frac{1}{2}\right\}}x^2$$
$$= \exp(4x)\frac{1}{\left\{D^2+9D+\frac{41}{2}\right\}}x^2 = 2\left[\frac{\exp(4x)}{41}\right]\left\{1-\frac{18}{41}D+\frac{242}{(41)^2}D^2\right\}x^2$$
$$= 2\left(\frac{\exp(4x)}{41}\right)\left\{x^2-\frac{36}{41}x+\frac{484}{(41)^2}\right\}.(6)$$

$$(7)\ I_{pi}: \frac{1}{\left(D+\frac{3}{2}\right)^2}[\exp(-x)\,2\,\cos(x)\,x]$$
$$= \exp\{x(-1-i)\}\frac{1}{\left(D-1-i+\frac{3}{2}\right)^2}x + \exp\{x(-1+i)\}\frac{1}{\left(D-1+i+\frac{3}{2}\right)^2}x$$
$$= \frac{\exp\{-x(1+i)\}}{-\frac{3}{4}-i}\left[1+\frac{4}{25}(3-4i)D\right]x + \frac{\exp\{-x(1-i)\}}{-\frac{3}{4}+i}\left[1+\frac{4}{25}(3+4i)D\right]x$$
$$= \left(\frac{8x\exp(-x)}{25}\right)[4\sin(x)-3\cos(x)] + \left(\frac{32\exp(-x)}{125}\right)[2\sin(x)+11\cos(x)]. \quad (7)$$

$$(8)\ I_{pi}: \ \frac{1}{\left(D-\frac{3}{2}\right)^2}[\exp(x)\,2\,\cos(x)\,x]$$

$$= \exp\{x(1+i)\}\frac{1}{\left(D+1+i-\frac{3}{2}\right)^2}x + \exp\{x(1-i)\}\frac{1}{\left(D+1-i-\frac{3}{2}\right)^2}x$$

$$= \frac{\exp\{x(1+i)\}}{-\frac{3}{4}-i}\left[1+\frac{4}{5}(1+2i)D\right]x + \frac{\exp\{x(1-i)\}}{-\frac{3}{4}+i}\left[1+\frac{4}{5}(1-2i)D\right]x$$

$$= -\left(\frac{8x\exp(x)}{25}\right)[4\sin(x)+3\cos(x)] + \left(\frac{8\exp(x)}{125}\right)[8\sin(x)-44\cos(x)]\,. \quad (8)$$

(9) $I_{pi}:\ \dfrac{1}{(D+2)^2}[\exp(-x)\,2i\ \sin(x)\,x]$

$$= \exp\{-x(1-i)\}\frac{1}{(D-1+i+2)^2}x - \exp\{-x(1+i)\}\frac{1}{(D-1-i+2)^2}x$$

$$= \frac{\exp\{-x(1-i)\}}{2i}[1+(i-1)D]x + \frac{\exp\{-x(1+i)\}}{2i}[1-(i+1)D]x$$

$$= i\exp(-x)[\sin(x)+\cos(x)-x\cos(x)]\ . \quad (9)$$

(10) $I_{pi}:\ \dfrac{1}{(D+1)^3}[\exp(x)\,2i\ \sin(x)\,x]$

$$= \exp\{x(1+i)\}\frac{1}{(D+1+i+1)^3}x - \exp\{x(1-i)\}\frac{1}{(D+1-i+1)^3}x$$

$$= \frac{\exp\{x(1+i)\}}{(2+i)^3}\left[1-\frac{3D}{2+i}\right]x - \frac{\exp\{x(1-i)\}}{(2-i)^3}\left[1-\frac{3D}{2-i}\right]x$$

$$= \frac{2i\exp(x)}{625}[21\sin(x)+72\cos(x)+10\,x\sin(x)-55\,x\cos(x)]\ . \quad (10)$$

(11) $I_{pi}:\ \dfrac{1}{D\left(D^2-4\right)}[4\cos^2(2x)\,x^2] = \dfrac{1}{D^3-4D}\left[\exp(4ix)+\exp(-4ix)+2\right]x^2$

$$= \exp(4ix)\frac{1}{(D+4i)^3-4(D+4i)}x^2 + \exp(-4ix)\frac{1}{(D-4i)^3-4(D-4i)}x^2$$

$$+\left[\frac{1}{-4D\left(1-\frac{D^2}{4}\right)}\right]2x^2 = \exp(4ix)\frac{1}{D^3+12iD^2-52D-80i}x^2$$

$$+\frac{\exp(-4ix)}{D^3-12iD^2-52D+80i}x^2 + \frac{\left(1-\frac{D^2}{4}\right)^{-1}}{-4D}2x^2$$

$$= \exp(4ix)\left[\frac{i}{80}-\frac{13}{1600}D-\frac{109i}{32000}D^2\right]x^2 +$$

$$\exp(-4ix)\left[\frac{-i}{80}-\frac{13}{1600}D+\frac{109i}{32000}D^2\right]x^2 - \frac{x}{4}-\frac{x^3}{6}$$

$$= -x^2\,\frac{\sin(4x)}{40} - 13\,x\,\frac{\cos(4x)}{400}$$

$$+\left(\frac{109}{8000}\right)\sin(4x)-\frac{x}{4}-\frac{x^3}{6} \quad . \tag{11}$$

$$(12)\ \ I_{pi}:\ \ \frac{1}{D^2\left(D^2-1\right)}[-4\sin^2(2x)x^2]=\frac{1}{D^4-D^2}\left[\exp(4ix)+\exp(-4ix)-2\right]x^2$$
$$=\exp(4ix)\frac{1}{(D+4i)^4-(D+4i)^2}x^2+\exp(-4ix)\frac{1}{(D-4i)^4-(D-4i)^2}x^2$$
$$+\left[\frac{1}{D^2\left(1-D^2\right)}\right]2x^2=$$
$$\exp(4ix)\times\frac{1}{D^4+16iD^3-97D^2-264iD+272}x^2$$
$$+\exp(-4ix)\times\frac{1}{D^4-16iD^3-97D^2+264iD+272}x^2$$
$$+\frac{1}{D^2}\left(1-D^2\right)^{-1}2x^2=$$
$$\frac{\exp(4ix)}{272}\left[1+\frac{33}{34}i\,D-\frac{2707}{4624}D^2\right]x^2$$
$$+\frac{\exp(-4ix)}{272}\left[1-\frac{33}{34}i\,D-\frac{2707}{4624}D^2\right]x^2+\frac{x^4}{6}+2x^2+4=$$
$$\frac{\cos(4x)}{136}\left(x^2-\frac{2707}{2312}\right)-\left(\frac{33}{17\times136}\right)x\,\sin(4x)$$
$$+\frac{x^4}{6}+2x^2+4. \qquad (12) \tag{3.91}$$

3.3.12 Problems Group V

On the right-hand side of the homogeneous linear ordinary differential equations with constant coefficients—see Problems Group I—additions have been made in the form $B(x)=\exp(\alpha x)\,W(x)$ where $W(x)\equiv\sin(x)x^n$, or $\cos(x)\,x^n$, etc. Work out their particular integral, I_{pi}.

$$\left(D^2+2D-3\right)u(x)=2\,\exp(2x)\,\cos(x)\,x\ .\ (1)$$
$$\left(D^2-3D-4\right)u(x)=2\,i\,\exp(2x)\,\sin(x)\,x\ .\ (2)$$
$$\left(D+\frac{1}{2}\right)^2u(x)=\sin(x)\,\cos(2x)\ .\ (3)$$
$$\left(D-\frac{1}{2}\right)^2u(x)=\sin(x)\,\sin(2x)\ .\ (4)$$
$$(D+2)^3\,u(x)=4\exp(x)\,\cos(x)\,x\ .\ (5)$$
$$(D-2)^3\,u(x)=4\exp(x)\,\sin(x)\,x\ .\ (6)$$
$$\left(D^2+D+1\right)u(x)=-4\,\exp(x)\,\sin^2(x)\,x\ .\ (7)$$

$$\left(D^2 - D + 1\right) u(x) = 4\, \exp(x)\, \cos^2(x)\, x \ . \quad (8)$$

$$\left(D^2 + 2D + 3\right) u(x) = 4\, x\, \cos(x)\, \exp(-x) \ . \quad (9)$$

$$\left(D^2 + 3D + 4\right) u(x) = -4\, x\, \sin(x)\, \exp(x) \ . \quad (10)$$

In particular, for problems (3) and (4), it is helpful to note the equalities $\sin(x)\ \cos(2\,x) = \sin(x) - 2\sin^3(x)$ and $\sin(x)\ \sin(2\,x) = 2\cos(x) - 2\cos^3(x)$.

3.4 Simultaneous Linear (ODEs) with Constant Coefficients

An (ODE) has only one independent variable and usually only a single dependent variable. But occasionally, the needs of a subject-matter require more than one dependent variable. For complete description then, there must be either partial differential equations or more than one simultaneous linear ordinary differential equation with constant coefficients. Here, we treat the latter option. Generally, the larger the number of these equations, the greater the effort needed to solve them. Therefore, for convenience, we work with equations that have only one, two, or three dependent variables. Because there are cases where the differential equations can be separated so that each is a function only of one dependent variable. Therefore, for additional convenience, we work with such cases first.

3.4.1 Separable Cases

Chose t as the single independent variable and notate a first differential with respect to t by the symbol Δ. That is

$$\Delta \equiv \frac{d}{\mathrm{d}t}.$$

{A} Number of Constants

{A}: Consider first a very simple example with only two dependent variables $y \equiv y(t)$ and $z \equiv z(t)$ that satisfy the following simultaneous equations.

$$\begin{aligned} \Delta y(t) \ - \ z(t) \ &= \ 0 \ , \\ y(t) \ - \ \Delta z(t) \ &= \ 0 \ . \end{aligned} \quad (3.92)$$

As noted in the passage following (3.4), the minimum number of constants needed for a complete solution to a single homogeneous linear ordinary differential equation with constant coefficients is equal to the order of the given differential equation. But when dealing with a series of coupled homogeneous linear ordinary differential equations with constant coefficients, the answer to this question is more subtle. Here, one needs to work with the determinant of the coefficients that multiply each of the

dependent variables and then look for the highest power of the differentials that occur. A practical demonstration helps with explanation of this statement.

The determinant of the operational coefficients of the dependent variables $y(t)$ and $z(t)$ in the two coupled equations (3.92) is

$$\begin{vmatrix} \Delta & -1 \\ 1 & -\Delta \end{vmatrix} \;=\; -\Delta^2 + 1 \,.$$

Because this determinant is not manifestly equal to zero, look for the highest power of Δ that occurs. It is equal to two. Therefore, the complete solution to this pair of differential equations will have two arbitrary constants.

{A} Solution

The dependent variables $y(t)$ and $z(t)$ in (3.92) can readily be separated. To this end, operate by Δ from the left on (3.92).

$$\Delta[\Delta y(t) - z(t)] \;=\; 0 \;;\;\; \Delta[y(t) - \Delta z(t)] \;=\; 0 \;. \tag{3.93}$$

Now, by using the original equation, (3.92), eliminate $-\Delta z(t)$ and $\Delta y(t)$.

$$\Delta[\Delta y(t)] - y(t) \;=\; 0 \;;\;\; \Delta[\Delta z(t)] - z(t) \;=\; 0 \;. \tag{3.94}$$

The result is two homogeneous linear ordinary differential equations with constant coefficients. Each of these two differential equations involves only a single dependent variable, that is, $y(t)$ or $z(t)$. And they both have constant coefficients. Now the well-worked procedure can be used to find their solution.

$$\begin{aligned} E_{ch;\,y} : k^2 - 1 \;=\; 0 \;:\; k_{1,2} \;&=\; \pm 1 \,;\; E_{ch;\,z} : k^2 - 1 \;=\; 0 \;:\; k_{1,2} \;=\; \pm 1. \\ S_{\text{comp};\,y}(t) \;&=\; \sigma_1 \exp(-t) + \sigma_2 \exp(t) \;;\; S_{\text{comp};\,z}(t) \\ &=\; \sigma_3 \exp(-t) + \sigma_4 \exp(t). \end{aligned} \tag{3.95}$$

Unfortunately, the full complementary solution, $S_{\text{comp};\,y}(t) + S_{\text{comp};\,z}(t)$, contains four arbitrary independent constants: $\sigma_1 \to \sigma_4$. But there should really be a total of only two undetermined constants. So what to do? Substitute the results, namely $y(t) = S_{\text{comp};\,y}(t)$ and $z(t) = S_{\text{comp};\,z}(t)$, into original differential equation (3.92) and see what happens. One gets

$$\Delta S_{\text{comp};\,y}(t) \;=\; S_{\text{comp};\,z}(t) \;;\; \Delta S_{\text{comp};\,z}(t) \;=\; S_{\text{comp};\,y}(t) \quad . \tag{3.96}$$

Good. This yields the required equalities

$$\begin{aligned} -\sigma_1 \exp(-t) + \sigma_2 \exp(t) \;&=\; \sigma_3 \exp(-t) + \sigma_4 \exp(t) \;; \\ -\sigma_3 \exp(-t) + \sigma_4 \exp(t) \;&=\; \sigma_1 \exp(-t) + \sigma_2 \exp(t) \;. \end{aligned} \tag{3.97}$$

And satisfaction of these equalities requires

$$\sigma_3 \;=\; -\sigma_1 \;;\; \sigma_4 \;=\; \sigma_2 \;. \tag{3.98}$$

Therefore, according to (3.95) and (3.98), the solution to the simultaneous linear ordinary differential equations with constant coefficients (3.92) is the following:

$$\begin{aligned} S_{\text{comp; }y}(t) \;&=\; \sigma_1 \exp(-t) + \sigma_2 \exp(t) \;; \\ S_{\text{comp; }z}(t) \;&=\; -\sigma_1 \exp(-t) + \sigma_2 \exp(t) \;. \end{aligned} \tag{3.99}$$

{B}: The simultaneous linear ordinary differential equations with constant coefficients {A} in (3.92) were easy to solve. Let us continue to treat similarly simple simultaneous equations but increase their number to three.

$$\begin{aligned} \Delta x(t) &= y(t) \quad ; \\ \Delta y(t) &= z(t) \quad ; \\ \Delta z(t) &= x(t) \quad . \end{aligned} \tag{3.100}$$

{B}: Number of Independent Constants

It is convenient to rewrite (3.100).

$$\begin{aligned} \Delta x(t) - y(t) + 0 \times z(t) \;&=\; 0 \;, \\ 0 \times x(t) + \Delta y(t) \;-\; z(t) \;&=\; 0 \;, \\ x(t) \;+ 0 \times y(t) - \;\Delta z(t) \;&=\; 0 \;. \end{aligned}$$

There are three dependent variables $x(t)$, $y(t)$, and $z(t)$ with t as the independent variable. The determinant of the operational coefficients of the given simultaneous equations is

$$\begin{vmatrix} \Delta & -1 & 0 \\ 0 & \Delta & -1 \\ 1 & 0 & \Delta \end{vmatrix} \;.$$

Again because this determinant is not manifestly equal to zero, the relevant parameter is the highest power of Δ that occurs. It is equal to three. Therefore, the complete solution will have three arbitrary constants.

{B}: Solution

Once again the dependent variables can readily be separated. To separate $x(t)$, from $y(t)$ and $z(t)$, operate by $\Delta\Delta$ from the left on the top equation in (3.100).

$$\Delta\Delta[\Delta x(t)] \;=\; \Delta\Delta y(t) \;. \tag{3.101}$$

Next operate by Δ on the middle equation in (3.100) and make use of the bottom equation there.

$$\Delta[\Delta y(t)] \;=\; \Delta z(t) \;=\; x(t)\;. \tag{3.102}$$

Combining (3.101) and (3.102) leads to a differential equation in single dependent variable $x(t)$. That is

$$(\Delta^3 - 1)x(t) = 0 \quad . \tag{3.103}$$

In fact, the same process can be repeated for the dependent variables $y(t)$ and $z(t)$. As a result, two remaining equations are obtained. Because of the $x(t)$, $y(t)$, $z(t)$ symmetry inherent in (3.100), the last two equations are very much like the first differential equation (3.103). That is

$$\begin{aligned}(\Delta^3 - 1)y(t) &= 0 \quad ;\\ (\Delta^3 - 1)z(t) &= 0 \quad .\end{aligned} \tag{3.104}$$

All these three homogeneous linear ordinary differential equations with constant coefficients are of third order. Therefore, complete solution of each will have three independent constants, thus making a grand total of nine constants. But as noted before, only three are needed. Therefore, six unnecessary constants will have to be eliminated. Fortunately, because the given simultaneous linear ordinary differential equations are all symmetric in structure, one needs to solve only one of the three. The $S_{\text{comp};\,x}(t)$ for (3.103) is found in the usual manner as follows.

$$E_{ch;\,x}\;:\;(k)^3 - 1 \;=\; 0\;:\;k_1 \;= 1\;;\;k_{2,3} = \frac{-1 \pm i\sqrt{3}}{2}\quad .$$

Therefore,

$$\begin{aligned}S_{\text{comp};\,x}(t) \;&=\; \sigma_1 \exp(t) + \exp\left(-\frac{t}{2}\right)\left\{\sigma_2 \exp\left(-i\,\frac{\sqrt{3}}{2}\,t\right) + \sigma_3 \exp\left(i\,\frac{\sqrt{3}}{2}\,t\right)\right\}\\ &= \sigma_1 \exp(t) + \exp\left(-\frac{t}{2}\right)\left\{(\sigma_2 + \sigma_3)\cos\left(\frac{\sqrt{3}}{2}t\right) - i\,(\sigma_2 - \sigma_3)\sin\left(\frac{\sqrt{3}}{2}t\right)\right\}.\end{aligned} \tag{3.105}$$

Because of symmetry, the other two S_{comp}s can be written by inspection.

$$\begin{aligned}S_{\text{comp};\,y}(t) \;&=\; \sigma_4 \exp(t) + \exp\left(-\frac{t}{2}\right)\left\{\sigma_5 \exp\left(-i\,\frac{\sqrt{3}}{2}\,t\right) + \sigma_6 \exp\left(i\,\frac{\sqrt{3}}{2}\,t\right)\right\}\\ &= \sigma_4 \exp(t) + \exp\left(-\frac{t}{2}\right)\left\{(\sigma_5 + \sigma_6)\cos\left(\frac{\sqrt{3}}{2}t\right) - i\,(\sigma_5 - \sigma_6)\sin\left(\frac{\sqrt{3}}{2}t\right)\right\};\\ S_{\text{comp};\,z}(t) \;&=\; \sigma_7 \exp(t) + \exp\left(-\frac{t}{2}\right)\left\{\sigma_8 \exp\left(-i\,\frac{\sqrt{3}}{2}\,t\right) + \sigma_9 \exp\left(i\,\frac{\sqrt{3}}{2}\,t\right)\right\}\\ &= \sigma_7 \exp(t) + \exp\left(-\frac{t}{2}\right)\left\{(\sigma_8 + \sigma_9)\cos\left(\frac{\sqrt{3}}{2}t\right) - i\,(\sigma_8 - \sigma_9)\sin\left(\frac{\sqrt{3}}{2}t\right)\right\}.\end{aligned} \tag{3.106}$$

To eliminate the six unnecessary constants, substitute the results provided in (3.105) and (3.106) in the form: $x(t) = S_{\text{comp; } x}(t)$, $y(t) = S_{\text{comp; } y}(t)$, and $z(t) = S_{\text{comp; } z}(t)$, into the original differential equations (3.100). One gets

$$\Delta S_{\text{comp; } x}(t) = S_{\text{comp; } y}(t) \; ; \tag{3.107}$$

$$\Delta S_{\text{comp; } y}(t) = S_{\text{comp; } z}(t) \; ; \tag{3.108}$$

$$\Delta S_{\text{comp; } z}(t) = S_{\text{comp; } x}(t) \; . \tag{3.109}$$

These equations provide the relationships that allow for the cancelation of six of the nine unknowns. But despite the straightforward nature of the needed algebra, direct elimination of these constants from (3.107) to (3.109), as they are currently structured, is a lengthy undertaking. With a view to finding a more convenient format for these equations that would help relieve this difficulty, rewrite $S_{\text{comp; } x}(t)$ in (3.105) as follows.

$$S_{\text{comp; } x}(t) = \sigma_1 \exp(t) + \sigma_0 \exp\left(-\frac{t}{2}\right) \times \left\{ \cos(\phi) \cos\left(\frac{\sqrt{3}}{2} t\right) - \sin(\phi) \sin\left(\frac{\sqrt{3}}{2} t\right) \right\} .$$

Or equivalently

$$S_{\text{comp; } x}(t) = \sigma_1 \exp(t) + \sigma_0 \exp\left(-\frac{t}{2}\right) \cos\left(\frac{\sqrt{3}}{2} t + \phi\right) . \tag{3.110}$$

Here,

$$\cos(\phi) = \left(\frac{\sigma_2 + \sigma_3}{\sigma_0}\right) \; ; \; \sin(\phi) = i \left(\frac{\sigma_2 - \sigma_3}{\sigma_0}\right) . \tag{3.111}$$

Because $\cos^2(\phi) + \sin^2(\phi) = 1$, the constant σ_0 is chosen such that

$$4 \left(\frac{\sigma_2 \sigma_3}{\sigma_0^2}\right) = 1 \; . \tag{3.112}$$

Note that $S_{\text{comp; } x}(t)$ still has three arbitrary constants. In addition to σ_1, there are the two new constants σ_0 and the angle ϕ. Of course, σ_0 and ϕ are functions of the original arbitrary constants σ_2 and σ_3. Again, because of symmetry already mentioned, the structure of $S_{\text{comp; } y}(t)$ and $S_{\text{comp; } z}(t)$ can be predicted by inspection.

$$S_{\text{comp; } y}(t) = \sigma_1' \exp(t) + \sigma_0' \exp\left(-\frac{t}{2}\right) \cos\left(\frac{\sqrt{3}}{2} t + \phi'\right) \; : \; (a)$$

$$S_{\text{comp; }z}(t) \;=\; \sigma_1'' \exp(t) + \sigma_0'' \exp\left(-\frac{t}{2}\right) \cos\left(\frac{\sqrt{3}}{2}t + \phi''\right) \;:\; (b). \quad (3.113)$$

At this juncture, get back to the original equations and work with them in the form (3.107)–(3.109). To this end, first use $S_{\text{comp; }x}(t)$ given in (3.110) and differentiate it to calculate $\Delta S_{\text{comp; }x}(t)$.

$$\begin{aligned}
\Delta S_{\text{comp; }x}(t) \;&=\; \sigma_1 \exp(t) \\
&+ \sigma_0 \exp\left(-\frac{t}{2}\right) \left\{-\frac{1}{2}\cos\left(\frac{\sqrt{3}}{2}t + \phi\right) - \frac{\sqrt{3}}{2}\sin\left(\frac{\sqrt{3}}{2}t + \phi\right)\right\}, \\
&= \sigma_1 \exp(t) + \sigma_0 \exp\left(-\frac{t}{2}\right) \cos\left(\frac{\sqrt{3}}{2}t + \phi + \frac{2\pi}{3}\right).
\end{aligned}$$

(Trigonometric equalities, $\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$ and $\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$, were used here.) Next write the result according to (3.107) as $S_{\text{comp; }y}(t) \;=\; \Delta S_{\text{comp; }x}(t)$. That is

$$\begin{aligned}
&\sigma_1' \exp(t) + \sigma_0' \exp\left(-\frac{t}{2}\right) \cos\left(\frac{\sqrt{3}}{2}t + \phi'\right) \\
&= \sigma_1 \exp(t) + \sigma_0 \exp\left(-\frac{t}{2}\right) \cos\left(\frac{\sqrt{3}}{2}t + \phi + \frac{2\pi}{3}\right).
\end{aligned}$$

This equation suggests the equalities:

$$\begin{aligned}
\sigma_1' &= \sigma_1 \;\;, \\
\sigma_0' &= \sigma_0 \;\;, \\
\phi' &= \phi + \frac{2\pi}{3} \;.
\end{aligned}$$

Accordingly,

$$S_{\text{comp; }y}(t) = \sigma_1 \exp(t) + \sigma_0 \exp\left(-\frac{t}{2}\right) \cos\left(\frac{\sqrt{3}}{2}t + \phi + \frac{2\pi}{3}\right). \quad (3.114)$$

Finally, because of symmetry, (3.110) and (3.114) lead by induction to the final result

$$S_{\text{comp; }z}(t) \;=\; \sigma_1 \exp(t) + \sigma_0 \exp\left(-\frac{t}{2}\right) \cos\left(\frac{\sqrt{3}}{2}t + \phi + \frac{4\pi}{3}\right). \quad (3.115)$$

{C}: The simultaneous differential equations in problem {B} were in principle elementary even though eliminating the six unnecessary independent constants took effort. In (3.116) given below, the level of complexity is raised a little. But here one is helped by the fact that there are fewer constants that need eliminating.

$$(\Delta + 1)x(t) - 2y(t) \;=\; \exp(t)\;;\;\; -x(t) + 2(\Delta + 1)y(t) \;=\; \exp(t)\,. \tag{3.116}$$

{C}: Number of Independent Constants

The independent variable is still t but now there are only two dependent variables $x(t)$ and $y(t)$. The determinant of the operational coefficients is

$$\begin{vmatrix} (\Delta + 1) & -2 \\ -1 & 2(\Delta + 1) \end{vmatrix}\,.$$

Because this determinant is not manifestly equal to zero, look for the highest power of Δ that occurs. Clearly, it is equal to two. Therefore, the complete solution to this pair of simultaneous linear ordinary differential equations will have only two arbitrary constants.

{C}: Solution

Following the usual protocol, it is necessary to remove one dependent variable from each of the given two simultaneous linear ordinary differential equations. Only elementary algebra is needed first to eliminate y and next x. The result is the following pair of differential equations.

$$\begin{aligned} (\Delta^2 + 2\Delta)x(t) &= 3\,\exp(t)\;; \\ 2(\Delta^2 + 2\Delta)y(t) &= 3\,\exp(t)\;. \end{aligned} \tag{3.117}$$

Using the established routine, (3.117) are straightforward to solve.

$$\begin{aligned} E_{ch;\,x}\,:\; k^2 + 2k \;&=\; 0\,;\; k_1 \;=0\,;\; k_2 = -2 \;\;; \\ E_{ch;\,y}\,:\; 2(k^2 + 2k) \;&=\; 0\,;\; k_1 \;=0\,;\; k_2 = -2 \;\;; \\ S_{\text{comp};\,x}(t) \;=\; \sigma_1 + \sigma_2\exp(-2\,t)\,;\;\; S_{\text{comp};\,y}(t) \;&=\; \sigma_3 + \sigma_4\exp(-2\,t) \;\;; \\ I_{pi\,;\,x} \;=\; \exp(t)\frac{1}{(\Delta+1)^2 + 2(\Delta+1)}.3 \;&=\; \exp(t)\frac{3}{3}\;; \\ I_{pi\,;\,y} \;=\; \exp(t)\frac{1}{2[(\Delta+1)^2 + 2(\Delta+1)]}.3 \;&=\; \exp(t)\frac{3}{6} \end{aligned} \tag{3.118}$$

Once again there are more independent constants—meaning $\sigma_1 \to \sigma_4$—than are needed. To eliminate the unnecessary additional constants, substitute the results for $x = S_{\text{comp};\,x}(t) + I_{pi\,;\,x}$ and $y = S_{\text{comp};\,y}(t) + I_{pi\,;\,y}$ into at least one of the original equations. The first of the original equation (3.116), namely $(\Delta + 1)\,x(t) - 2\,y(t) = \exp(t)$, gives

$$(\Delta + 1)\left\{I_{pi\,;\,x} + S_{\text{comp};\,x}(t)\right\} - 2\left\{I_{pi\,;\,y} + S_{\text{comp};\,y}(t)\right\} = \exp(t)\ . \quad (3.119)$$

That is

$$\begin{aligned}&(\Delta + 1)\left\{\exp(t) + \sigma_1 + \sigma_2 \exp(-2\,t)\right\}\\ -2&\left\{\frac{1}{2}\exp(t) + \sigma_3 + \sigma_4 \exp(-2\,t)\right\} = \exp(t)\ .\end{aligned} \quad (3.120)$$

Or equivalently

$$(\sigma_1 - 2\sigma_3) - (\sigma_2 + 2\sigma_4)\exp(-2\,t) = 0\ . \quad (3.121)$$

For arbitrary t, this equation can be satisfied only if $\sigma_3 = \frac{\sigma_1}{2}$ and $\sigma_4 = -\frac{\sigma_2}{2}$. Hence, the complete solution of the simultaneous differential equations (3.116) is

$$\begin{aligned}x(t) &= \sigma_1 + \sigma_2 \exp(-2\,t) + \exp(t)\ ;\\ y(t) &= \frac{\sigma_1}{2} - \frac{\sigma_2}{2}\exp(-2\,t) + \frac{1}{2}\exp(t)\ .\end{aligned} \quad (3.122)$$

{D}: Simultaneous linear differential equations (3.123) given below will be solved next.

$$\begin{aligned}(\Delta + 1)^2 x(t) + (\Delta + 2)^2 y(t) &= \exp(t)\ ;\\ (\Delta + 1)x(t) + (\Delta + 2)y(t) &= \exp(t)\,t\ .\end{aligned} \quad (3.123)$$

{D}: Number of Independent Constants

Again, there are two dependent variables x and y and t is the independent variable. The determinant of the operational coefficients is

$$\begin{vmatrix}(\Delta + 1)^2 & (\Delta + 2)^2\\ (\Delta + 1) & (\Delta + 2)\end{vmatrix}\ .$$

Because this determinant is not manifestly equal to zero, look for the highest power of Δ that occurs. It is equal to two. Therefore, the complete solution to this pair of differential equations will have two arbitrary constants.

{D}: Solution

To eliminate $x(t)$, multiply the bottom line in (3.123) by $(\Delta + 1)$ and subtract the result from the line above.

$$(\Delta + 2)^2 y(t) - (\Delta + 1)(\Delta + 2)y(t) \;=\; \exp(t) - (\Delta + 1)\,\exp(t)\,t\ . \quad (3.124)$$

Equivalently,

$$(\Delta + 2)y(t) = -2\,t\ \exp(t)\ . \tag{3.125}$$

Following the usual routine, (3.125) gives:

$$\begin{aligned} E_{ch;\,y} : k + 2 = 0\ ;\ k_1 = -2\ ;\ S_{\text{comp};\,y}\ &=\ \sigma_1 \exp(-2\,t)\ ; \\ I_{pi\,;\,y}\ =\ \frac{1}{(\Delta) + 2}\exp(t)(-2\,t)\ &=\ \exp(t)\frac{1}{(\Delta + 1) + 2}(-2\,t) \\ =\ \exp(t)\frac{1}{3}\left(1 - \frac{\Delta}{3}\right)(-2\,t)\ &=\ \frac{\exp(t)}{3}\left(-2\,t + \frac{2}{3}\right)\ . \end{aligned} \tag{3.126}$$

Clearly, σ_1 is one of the two independent constants. This leaves only one additional independent constant to find.

Next, in order to eliminate $y(t)$, multiply the bottom line in (3.123) by $(\Delta + 2)$ and subtract the result from the top line.

$$(\Delta + 1)^2 x9t) - (\Delta + 2)(\Delta + 1)x = \exp(t)\ -(\Delta + 2)\,[\exp(t)\,t]\ .$$

Equivalently,

$$(\Delta + 1)x(t) = 3\,t\ \exp(t)\ . \tag{3.127}$$

Again following the usual routine, one gets

$$\begin{aligned} E_{ch;\,x} : k + 1 = 0\ ;\ k_1 = -1\ ;\ S_{\text{comp};\,x}(t)\ &=\ \sigma_2 \exp(-t)\ ; \\ I_{pi\,;\,x}\ =\ \frac{1}{(\Delta) + 1}\exp(t)(3\,t)\ &=\ \exp(t)\frac{1}{(\Delta + 1) + 1}(3\,t) \\ =\ \exp(t)\frac{1}{2}\left(1 - \frac{\Delta}{2}\right)(3\,t)\ &=\ \frac{\exp(t)}{2}\left(3\,t - \frac{3}{2}\right)\ . \end{aligned} \tag{3.128}$$

{E}: Solve

$$\begin{aligned} (\Delta^2 + 3\Delta + 4)x(t) + (\Delta^2 + \Delta + 4)y(t) &= \exp(t)\,t\ , \\ (2\Delta + 1)x(t) + (\Delta + 1)y(t) &= 2\,t\ . \end{aligned} \tag{3.129}$$

{E}: Number of Independent Constants

Again, t is the independent variable and there are two dependent variables $x(t)$ and $y(t)$. The determinant of their operational coefficients is

$$\begin{vmatrix} (\Delta^2 + 3\Delta + 4) & (\Delta^2 + \Delta + 4) \\ (2\Delta + 1) & (\Delta + 1) \end{vmatrix}\ .$$

Because this determinant is not manifestly equal to zero, look for the highest power of Δ that occurs. It is equal to three. Therefore, the complete solution to the simultaneous linear ordinary differential equation (3.129) will have three arbitrary constants.

{E}: Solution

To eliminate $x(t)$, multiply the top line in (3.129) by $(2\Delta + 1)$ and the bottom by $(\Delta^2 + 3\Delta + 4)$. Subtract the resultant equation at the bottom from the corresponding one at the top.

$$\Delta(\Delta^2 - \Delta + 2)y(t) = (2\Delta + 1)\exp(t)\,t - (\Delta^2 + 3\Delta + 4)\{2\,t\} \ .$$

Equivalently,

$$\Delta(\Delta^2 - \Delta + 2)y(t) = 3\exp(t)\,t + 2\exp(t) - 8\,t - 6 \ . \tag{3.130}$$

Following the usual routine, (3.130) gives:

$$\begin{aligned}
E_{ch;\,y} : \qquad & k(k^2 - k + 2) = 0\ ;\ k_1 = 0\ ,\ k_{2,3} = \frac{1 \pm i\sqrt{7}}{2}\ ; \\
S_{\text{comp};\,y}(t) \ &= \ \sigma_1 + \exp\left(\frac{t}{2}\right)\left\{\sigma_2 \sin\left(\frac{\sqrt{7}}{2}t\right) + \sigma_3 \cos\left(\frac{\sqrt{7}}{2}t\right)\right\}\ ; \\
I_{pi\,;\,y} \ &= \ \frac{1}{\Delta(\Delta^2 - \Delta + 2)}\{\exp(t)(3\,t + 2) - 8\,t - 6\} \\
&= \ \exp(t)\frac{1}{(\Delta + 1)[(\Delta + 1)^2 - (\Delta + 1) + 2]}\{3\,t + 2\} \\
&\quad -\frac{1}{\Delta(\Delta^2 - \Delta + 2)}(8\,t + 6) \\
&= \exp(t)\left[\frac{1}{2}\left(1 - \frac{3}{2}\Delta\right)\{3\,t + 2\}\right] - \frac{1}{2}\left\{\frac{1}{\Delta} + \frac{1}{2} - \frac{\Delta}{2}\right\}(8\,t + 6) \ . \\
I_{pi\,;\,y} \ &= \ \frac{\exp(t)}{2}\left(3\,t - \frac{5}{2}\right) - 2\,t^2 - 5\,t + \frac{1}{2} \qquad .
\end{aligned} \tag{3.131}$$

Similarly, to eliminate $y(t)$ multiply the top line in (3.129) by $(\Delta + 1)$ and the bottom by $(\Delta^2 + \Delta + 4)$. Subtract the resultant equation at the top from the corresponding one at the bottom.

$$\begin{aligned}
\Delta(\Delta^2 - \Delta + 2)x(t) \ &= \ (\Delta^2 + \Delta + 4)\{2\,t\} - (\Delta + 1)\exp(t)\,t \ , \\
&= -2\exp(t)\,t - \exp(t) + 8\,t + 2 \ .
\end{aligned} \tag{3.132}$$

Equivalently,

$$\Delta(\Delta^2 - \Delta + 2)x = -2\exp(t)\,t - \exp(t) + 8\,t + 2. \tag{3.133}$$

According to the well-established routine, (3.133) gives

$$
\begin{aligned}
E_{ch;\,x}: \qquad & k(k^2-k+2)=0\,;\ k_1=0\,,\ k_{2,3}=\frac{1\pm i\sqrt{7}}{2}\,;\\
S_{\text{comp};\,x}(t) \;&=\; \sigma_4+\exp\left(\frac{t}{2}\right)\left\{\sigma_5\sin\left(\frac{\sqrt{7}}{2}t\right)+\sigma_6\cos\left(\frac{\sqrt{7}}{2}t\right)\right\}\,;\\
I_{pi\,;\,x} \;&=\; \frac{1}{\Delta(\Delta^2-\Delta+2)}\{\exp(t)(-2\,t-1)+8\,t+2\}\\
&=\; \exp(t)\left[\frac{1}{(\Delta+1)[(\Delta+1)^2-(\Delta+1)+2]}\right]\{-2\,t-1\}\\
&\quad+\left\{\frac{1}{\Delta(\Delta^2-\Delta+2)}\right\}(8\,t+2)\\
&=\exp(t)\left[\frac{1}{2}\left(1-\frac{3}{2}\Delta\right)\{-2\,t-1\}\right]+\left\{\frac{1}{2\Delta}+\frac{1}{4}-\frac{\Delta}{4}\right\}(8\,t+2)\ .\\
I_{pi\,;\,x} \;&=\; \exp(t)\,(1-t)+2\,t^2+3\,t-\frac{3}{2}\ .
\end{aligned}
\tag{3.134}
$$

As usual there are more independent constants, meaning six—that is, $\sigma_1 \to \sigma_6$—instead of the only three that are needed. And again, to eliminate the unnecessary additional constants, substitute the results for $x(t) = S_{\text{comp};\,x}(t) + I_{pi\,;\,x}$ and $y(t) = S_{\text{comp};\,y}(t) + I_{pi\,;\,y}$—that is

$$
\begin{aligned}
x(t) \;&=\; \exp(t)\,(1-t)+2\,t^2+3\,t-\frac{3}{2}\\
&+\;\sigma_4+\exp\left(\frac{t}{2}\right)\left\{\sigma_5\sin\left(\frac{\sqrt{7}}{2}t\right)+\sigma_6\cos\left(\frac{\sqrt{7}}{2}t\right)\right\}\ ;\\
y(t) \;&=\; \frac{\exp(t)}{2}\left(3\,t-\frac{5}{2}\right)-2\,t^2-5\,t+\frac{1}{2}\\
&+\;\sigma_1+\exp\left(\frac{t}{2}\right)\left\{\sigma_2\sin\left(\frac{\sqrt{7}}{2}t\right)+\sigma_3\cos\left(\frac{\sqrt{7}}{2}t\right)\right\}\ ,
\end{aligned}
\tag{3.135}
$$

into at least one of the original equations. The second of the original equation (3.123)—namely $(2\Delta+1)\,x(t)+(\Delta+1)y(t)\ -\ 2\,t\ =\ 0$—is clearly the easier to handle. After a little algebra, one gets

$$
\begin{aligned}
&(\sigma_1+\sigma_4)+\exp\left(\frac{t}{2}\right)\sin\left(\frac{\sqrt{7}}{2}t\right)\left\{\frac{3}{2}(\sigma_2+\sigma_5)-\frac{\sqrt{7}}{2}(\sigma_3+\sigma_6)\right\}\\
+\;&\exp\left(\frac{t}{2}\right)\cos\left(\frac{\sqrt{7}}{2}t\right)\left\{\frac{3}{2}(\sigma_3+\sigma_6)+\frac{\sqrt{7}}{2}(\sigma_2+\sigma_5)\right\}\ =\ 0\ .
\end{aligned}
\tag{3.136}
$$

For arbitrary t, this equation can be satisfied only if

$$(\sigma_1 + \sigma_4) \;=\; 0\,;\;\; (\sigma_2 + \sigma_5) \;=\; 0\,;\;\; (\sigma_3 + \sigma_6) \;=\; 0 \quad . \tag{3.137}$$

Insert $\sigma_4 = -\sigma_1,\ \sigma_5 = -\sigma_2$ and $\sigma_6 = -\sigma_3$ into (3.135) describing $x(t)$. One gets

$$\begin{aligned} x(t) \;&=\; \exp(t)\,(1-t) + 2\,t^2 + 3\,t - \frac{3}{2} \\ &-\; \sigma_1 - \exp\left(\frac{t}{2}\right)\left\{\sigma_2 \sin\left(\frac{\sqrt{7}}{2}t\right) + \sigma_3 \cos\left(\frac{\sqrt{7}}{2}t\right)\right\} \quad . \end{aligned} \tag{3.138}$$

Equations (3.135) for $y(t)$ and (3.138) for $x(t)$ are the complete solution to the pair of simultaneous linear ordinary differential equation (3.129) and appropriately have only three arbitrary constants $\sigma_1,\ \sigma_2$, and σ_3.

{F}: Solve

$$\begin{aligned} (\Delta)x(t) + (4\,\Delta)y(t) &= \sigma_0\, t \quad , \\ (3\,\Delta)x(t) + (12\,\Delta)y(t) &= 2\,t \quad , \end{aligned} \tag{3.139}$$

where t is the independent variable and $x(t)$ and $y(t)$ are dependent variables.

{F}: Number of Independent Constants

The determinant of the operational coefficients of (3.139) is

$$\begin{vmatrix} (\Delta) & (4\Delta) \\ (3\Delta) & (12\Delta) \end{vmatrix} = 0. \tag{3.140}$$

In other words, the relevant determinant is manifestly equal to zero. Thus, one wonders as to whether a solution to the simultaneous linear ordinary differential equation (3.139) is at all possible? And, if a solution is possible, how many independent constants would there be?

In an attempt to deal with these issues, let us separate the equations. To this end, multiply the first of the (3.139) from the left with (12Δ) and the second with (4Δ) and subtract the second from the first, thereby canceling the terms multiplying $y(t)$. One gets

$$\begin{aligned} [(12\Delta) \times (\Delta) - (4\Delta) \times (3\Delta)]x(t) = 0 \;&=\; (12\Delta)(\sigma_0\, t) - (4\Delta)(2\,t) \\ &= 12\,\sigma_0 - 8 \end{aligned} \tag{3.141}$$

Therefore, in order that there be a solution to (3.139), the following has to be true

$$\sigma_0 = \frac{2}{3}\,. \tag{3.142}$$

And if this relationship should be true, then the two equations in (3.139) are identical and one is left with only a single equation: namely $(\Delta)x(t) + (4\,\Delta)y(t) = \left(\frac{2}{3}\right)\,t$. And by itself this equation contains insufficient information to solve for both $x(t)$ and $y(t)$. On the other hand, if one of the independent variables—say $y(t)$—was chosen, the differential equation involving $x(t)$ would very likely have a solution. But because the choice of $y(t)$ is arbitrary, it should in principle have an arbitrary number of constants. Thus, there is no fixed limit as to how many independent constants such a solution would have.

3.4.2 *Problems Group VI*

Solve the following pairs of simultaneous linear ordinary differential equations with constant coefficients. Reminder: $\Delta = \frac{d}{\mathrm{d}t}$ and $D = \frac{d}{\mathrm{d}x}$.

$$
\begin{aligned}
(\Delta+1)x(t) + (\Delta+2)y(t) &= \exp(t) && ,\\
(2\Delta+1)x(t) + (4\Delta+3)y(t) &= \exp(t)\,t && . \quad (1)\\
(2\Delta+1)x(t) + (\Delta+2)y(t) &= \exp(t)\,t && ,\\
(\Delta+2)x(t) + (2\Delta+1)y(t) &= t && . \quad (2)\\
(\Delta^2+\Delta+1)x(t) + (\Delta+1)y(t) &= t && ,\\
(\Delta^2+\Delta)x(t) + (\Delta+2)y(t) &= t\,\exp(t) && . \quad (3)\\
[(-x+3)D+1]\,u(x) + [(-x+3)D+2]\,v(x) &= -\frac{1}{x-3} && ,\\
[2(-x+3)D+1]\,u(x) + [4(-x+3)D+3]\,v(x) &= \frac{1}{x-3}\log(-x+3) && . \quad (4)\\
[2(2x-1)D+1]\,u(x) + [(2x-1)D+2]\,v(x) &= \frac{1}{2}\log(2x-1)\sqrt{(2x-1)} && ,\\
[(2x-1)D+2]\,u(x) + [2(2x-1)D+1]\,v(x) &= \frac{1}{2}\log(2x-1) && . \quad (5)\\
\left[(x+1)^2D^2 + 2(x+1)D + 1\right]u(x) + [(x+1)D+1]\,v(x) &= \log(x+1) && ,\\
\left[(x+1)^2D^2 + 2(x+1)D\right]u(x) + [(x+1)D+2]\,v(x) &= (x+1)\log(x+1) && . \quad (6)
\end{aligned}
\tag{3.143}
$$

Chapter 4
Variable Coefficients



Both single and simultaneous linear ordinary differential equations with constant coefficients were discussed in detail in Chap. 3. Here, that analysis is extended to linear ordinary differential equations with variable coefficients. Differential equations with non-constant coefficients are in general much harder to solve. For simplicity, therefore, only first-order and first-degree equations of type (A) are handled.

$$(A) \;:\; Du(x) + M(x)u(x) \;=\; N(x) \;.$$

These equations are treated by using an appropriate integrating factor.

Included also is an extensive discussion of equations which outwardly look fearsome: that is, equations of type

$$Du(x) + M(x)u(x) \;=\; N(x)u^n(x) \;.$$

But with help from the $Bernouilli^{1.}$ suggestion, these equations can be transformed into equations similar to (A).

4.1 Linear (ODE)'s

4.1.1 First-Order and First-Degree

Linear ordinary differential equations that are of the first-order and first-degree and have variable coefficients are of the general form

$$f_1(x)Du(x) + f_2(x)u(x) = f_3(x) \;. \tag{4.1}$$

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More conveniently, they are written as

$$Du(x) + M(x)u(x) = N(x) \tag{4.2}$$

where $N(x) = \frac{f_3(x)}{f_1(x)}$ and $N(x) = \frac{f_2(x)}{f_1(x)}$ and $D = \frac{\mathrm{d}}{\mathrm{d}x}$.

4.1.2 Integrating Factor

Multiply (on the left) the differential equation (4.2) by an as yet unknown factor $F(x)$.

$$F(x)[Du(x) + M(x)\,u(x)] = F(x)N(x)\ . \tag{4.3}$$

Assume $F(x)$ is so chosen that it admits of the following equality

$$F(x)[Du(x) + M(x)\,u(x)] \ = \ D[F(x)\,u(x)]\ . \tag{4.4}$$

This implies

$$F(x)\,[Du(x)] + [F(x)\,M(x)]\,u(x) \ = \ F(x)\,[Du(x)] + u(x)\,[DF(x)]\ . \tag{4.5}$$

Remove $F(x)[Du(x)]$ from both sides of the relationship (4.5).

$$[F(x)\,M(x)]\,u(x) = u(x)\,[DF(x)]\ . \tag{4.6}$$

Notice the function $u(x)$ is not operated upon either side of (4.6). Indeed, it appears simply as a multiplying factor. Therefore, it can be eliminated. The result is a differential equation obeyed by the as yet unknown factor $F(x)$.

$$F(x)\,M(x) = DF(x)\ . \tag{4.7}$$

Multiply (from the left) by $\frac{1}{F(x)}$ and integrate with respect to x.

$$\begin{aligned} &\int \frac{1}{F(x)} F(x)M(x)\,.\,\mathrm{d}x \ = \ \int M(x)\,.\,\mathrm{d}x \\ \equiv &\int \frac{1}{F(x)}\,.\,DF(x)\,.\,\mathrm{d}x \ = \ \int \frac{1}{F(x)}\,.\,\mathrm{d}F(x) \ = \ \log F(x) - \log \sigma_0\ . \end{aligned}$$

Or equivalently

$$\log F(x) = \log \sigma_0 + \int M(x)\,.\,\mathrm{d}x\ . \tag{4.8}$$

Exponentiation determines the factor $F(x)$.

$$F(x) = \sigma_0\, W(x) \ , \tag{4.9}$$

where,

$$W(x) = \exp\left(\int M(x)\,\mathrm{d}x\right) \tag{4.10}$$

and σ_0 is an arbitrary constant. Henceforth, $W(x)$ will be referred to as the integrating factor. Clearly, $W(x)$ is a known function because it depends only on $M(x)$ which is available as part of the relevant differential equation itself. [For instance, see (4.2).]

4.1.3 Equation (4.2): Solution

Left-hand sides of (4.3) and (4.4) are the same. Therefore, their right-hand sides must be equal.

$$F(x)N(x) \ = \ D[F(x)\,u(x)] \ . \tag{4.11}$$

In (4.11), as per (4.9), replace $F(x)$ by $\sigma_0 W(x)$.

$$\sigma_0 W(x)N(x) \ = \ \sigma_0 D[W(x)\,u(x)] \ . \tag{4.12}$$

Remove σ_0 from both sides and integrate with respect to x .

$$\int W(x)\,N(x)\,\mathrm{d}x \ = \ \int D[W(x)\,u(x)]\,\mathrm{d}x \ . \tag{4.13}$$

Work through the right-hand side of (4.13).

$$\int \frac{\mathrm{d}[W(x)\,u(x)]}{\mathrm{d}x}\,\mathrm{d}x \ = \ \int \mathrm{d}[W(x)\,u(x)] \ = \ W(x)\,u(x) - \sigma_1 \ .$$

Equate the left- and the right-hand sides of (4.13),

$$\int W(x)\,N(x)\,\mathrm{d}x \ = \ W(x)\,u(x) - \sigma_1 \ , \tag{4.14}$$

transfer σ_1 and divide the result by $W(x)$. Voila !

We have the solution $u(x)$ of the differential equation (4.3).

$$u(x) = \frac{1}{W(x)}\left[\int W(x)\,N(x)\,\mathrm{d}x + \sigma_1\right] \ . \tag{4.15}$$

The unknown constant σ_1 can be determined by a single boundary condition.

Note

Given below are examples group I and problems group I. They both retain the notation $M(x)$ and $N(x)$ as in (4.2), and $W(x)$ as in (4.10). The solution, $u(x)$, is provided by (4.15).

4.2 Examples Group I

Use integrating factor $W(x)$ and solve the following twelve differential equations. The notation is as in (4.2), (4.10), and (4.15).

$$
\begin{aligned}
(1): &\quad M(x) = \frac{3}{x} \;,\; N(x) = x^2 \;. \\
(2): &\quad M(x) = 2x \;,\; N(x) = x^3 \;. \\
(3): &\quad M(x) = \frac{3+x}{x} \;,\; N(x) = \frac{2}{x} \;. \\
(4): &\quad M(x) = \frac{1}{x} \;,\; N(x) = x^2 \exp(x) \;. \\
(5): &\quad M(x) = 3\tan(x) \;,\; N(x) = 2\sec(x) \;. \\
(6): &\quad M(x) = 3\cot(x) \;,\; N(x) = 2\sin(2x) \;. \\
(7): &\quad M(x) = \frac{2x}{x-2} \;,\; N(x) = 5\exp(-x) \;. \\
(8): &\quad M(x) = -\frac{2}{x} \;,\; N(X) = x^3 \exp(-x) \;. \\
(9): &\quad M(x) = -\frac{2}{x+1} \;,\; N(x) = x-1 \;. \\
(10): &\quad M(x) = -\frac{2}{x-1} \;,\; N(x) = (x-1)^2 \;. \\
(11): &\quad M(x) = \frac{1}{x\log(x)} \;,\; N(x) = x^4 \;. \\
(12): &\quad M(x) = \frac{1}{x} + \cot(x) \;,\; N(x) = \cot(x) \;.
\end{aligned}
\tag{4.16}
$$

4.2.1 Solution

Use (4.16)-(1)–(4.16)-(12): First, work out $W(x)$ according to the procedure described in (4.10). Next, use (4.15) to determine the solution to the relevant differential equation. [Note: In problem (2) above, the integral $\int x^3 \exp(x^2)\,\mathrm{d}x$ can be

done as follows: Set $x^2 = y$. Then, $2x\,\mathrm{d}x = \mathrm{d}y$ and the integral can be written as $\frac{1}{2}\int y\exp(y)\,\mathrm{d}y = \frac{1}{2}\exp(y)(y-1) = \exp(x^2)\frac{(x^2-1)}{2}$.]

$$(1):\quad W(x) = \exp\left[\int\left(\frac{3}{x}\right)\mathrm{d}x\right] = \exp\left[\log(x^3)\right] = x^3\,;$$

$$u = \frac{1}{x^3}\left[\int x^3\times x^2\,\mathrm{d}x+\sigma_0\right] = \frac{1}{x^3}\left[\frac{x^6}{6}+\sigma_0\right]\,.$$

$$(2):\quad W(x) = \exp\left[\int 2x\,\mathrm{d}x\right] = \exp(x^2)\,;$$

$$u = \frac{1}{\exp(x^2)}\left(\int x^3\,\exp(x^2)\,\mathrm{d}x+\sigma_0\right) = \left(\frac{x^2-1}{2}\right)+\sigma_0\exp(-x^2)\,.$$

$$(3):\ W(x) = \exp\left[\int\left(\frac{3+x}{x}\right)\mathrm{d}x\right] = \exp[3\log(x)+x] = x^3\exp(x)\,;$$

$$u = \frac{1}{x^3\exp(x)}\left(\int 2x^2\exp(x)\,\mathrm{d}x+\sigma_0\right) = 2\left(\frac{x^2-2x+2}{x^3}\right)+\frac{\sigma_0}{x^3\exp(x)}\,.$$

$$(4):\quad W(x) = \exp\left[\int\frac{1}{x}\,\mathrm{d}x\right] = \exp\left[\log(x)\right] = x\,;$$

$$u = \frac{1}{x}\left(\int x^3\,\exp(x)\,\mathrm{d}x+\sigma_0\right) = \frac{1}{x}[\exp(x)(x^3-3x^2+6x-6)+\sigma_0]\,.$$

$$(5):\quad W(x) = \exp\left[\int 3\tan x\,\mathrm{d}x\right] = \exp[-3\log(\cos x)] = \frac{1}{(\cos x)^3}\,;$$

$$u = (\cos x)^3\left(\int\frac{2}{(\cos x)^3}\,\sec x\,\mathrm{d}x+\sigma_0\right)$$

$$= (\cos x)^3\left[\frac{(\sec x)^3}{3}(3\sin x+\sin 3x)+\sigma_0\right] = \frac{(3\sin x+\sin 3x)}{3}+\sigma_0(\cos x)^3.$$

$$(6):\quad W(x) = \exp\left[\int 3\cot x\,\mathrm{d}x\right] = \exp\left[3\log(\sin x)\right] = (\sin x)^3\,;$$

$$u = \frac{1}{(\sin x)^3}\left(\int(\sin x)^3\,2\sin(2x)\,\mathrm{d}x+\sigma_0\right)$$

$$= \frac{1}{(\sin x)^3}\left[\frac{10\sin x-5\sin(3x)+\sin(5x)}{20}+\sigma_0\right]\,.$$

$$(7):\quad W(x) = \exp\left[\int\frac{2x}{x-2}\,\mathrm{d}x\right] = \exp\left[\int 2\left(1+\frac{2}{x-2}\right)\mathrm{d}x\right]$$

$$= \exp\left[2x+4\log(x-2)\right] = (x-2)^4\exp(2x)\,;$$

$$u = \frac{1}{(x-2)^4\exp(2x)}\left[\int 5(x-2)^4\exp(x)\,\mathrm{d}x+\sigma_0\right]$$

$$= \frac{1}{(x-2)^4\exp(2x)}\left[5(x^4-12x^3+60x^2-152x+168)\exp(x)+\sigma_0\right]\,.$$

$$(8): \quad W(x) = \exp\left[\int \frac{-2}{x}\,\mathrm{d}x\right] = \exp[-2\log(x)] = \frac{1}{\exp[\log(x^2)]} = \frac{1}{x^2} \ ;$$

$$u = x^2\left(\int \left(\frac{1}{x^2}\right) x^3 \exp(-x)\,\mathrm{d}x + \sigma_0\right) = x^2\left[-\exp(-x)\,(x+1) + \sigma_0\right] .$$

$$(9): \quad W(x) = \exp\left[\int \frac{-2}{x+1}\,\mathrm{d}x\right] = \exp\left[-2\log(x+1)\right] = \frac{1}{(x+1)^2} \ ;$$

$$u = (x+1)^2\left[\int \left(\frac{(x-1)}{(x+1)^2}\right)\mathrm{d}x + \sigma_0\right]$$

$$= (x+1)^2\left[\frac{2}{(x+1)} + \log(1+x) + \sigma_0\right] \quad .$$

$$(10): \quad W(x) = \exp\left[\int \frac{-2}{x-1}\,\mathrm{d}x\right] = \exp[-2\log(x-1)] = \frac{1}{(x-1)^2} \ ;$$

$$u = (x-1)^2\left(\int \frac{(x-1)^2}{(x-1)^2}\,\mathrm{d}x + \sigma_0\right) = (x-1)^2\,[x + \sigma_0] \ .$$

$$(11): \quad W(x) = \exp\left[\int \left(\frac{1}{x\,\log(x)}\right)\mathrm{d}x\right] = \exp\int\left[\frac{1}{\log x}\mathrm{d}(\log x)\right]$$

$$= \exp\left[\log(\log x)\right] = \log(x) \ ;$$

$$u = \left(\frac{1}{\log(x)}\right)\left[\int x^4 \log(x)\,\mathrm{d}x + \sigma_0\right] = \left(\frac{1}{\log(x)}\right)\left[\frac{x^5}{5}\log(x) - \frac{x^5}{25} + \sigma_0\right].$$

$$(12): \quad W(x) = \exp\left[\int \left(\frac{1}{x} + \cot x\right)\mathrm{d}x\right] = \exp\left[\log(x) + \log(\sin x)\right]$$

$$= \exp\left[\log(x\,\sin x\right] = x\,\sin x \quad ;$$

$$u = \left(\frac{1}{x\,\sin x}\right)\left[\int x\,\sin x\,\cot x\,\mathrm{d}x + \sigma_0\right] = \left(\frac{1}{x\,\sin x}\right)\left[\int x\,\cos x\,\mathrm{d}x + \sigma_0\right]$$

$$= \left(\frac{1}{x\,\sin x}\right)[x\,\sin x + \cos x + \sigma_0] \ . \tag{4.17}$$

4.2.2 *Problems Group I*

For given choices of the duo $M(x)$ and $N(x)$, solve the following differential equations labeled (1)–(12). [Hint: Compare (4.2), (4.10), and (4.15)–(4.17).]

$$(1): \qquad M(x) = \frac{1}{x} \quad , \quad N(x) = x \ .$$

$$(2): \qquad M(x) = \frac{3}{x} \quad , \quad N(x) = x^2 \ .$$

$$
\begin{aligned}
(3): \quad & M(x) = \frac{2+x}{x} \quad , \quad N(x) = \frac{3}{x} \ . \\
(4): \quad & M(x) = \frac{1+3x}{x} \quad , \quad N(x) = x\exp(x) \ . \\
(5): \quad & M(x) = \cot x \quad , \quad N(x) = \sec x \ . \\
(6): \quad & M(x) = \tan x \quad , \quad N(x) = \cos x \ . \\
(7): \quad & M(x) = \frac{x}{x-1} \quad , \quad N(x) = x\exp(x) \ . \\
(8): \quad & M(x) = 3\cot x \quad , \quad N(X) = 2\cos x \ . \\
(9): \quad & M(x) = -\frac{1}{x+1} \quad , \quad N(x) = (x+1)^2 \ . \\
(10): \quad & M(x) = -\frac{1}{x-1} \quad , \quad N(x) = (x-1)^2 \ . \\
(11): \quad & M(x) = \frac{2}{x\log(x)} \quad , \quad N(x) = x \ . \\
(12): \quad & M(x) = \frac{1}{x} + \cot x \quad , \quad N(x) = \cot x \ .
\end{aligned}
\tag{4.18}
$$

4.3 Bernouilli Equation

When $n = 0$, the following equation

$$Du(x) + M(x)u(x) = N(x)u^n(x) \tag{4.19}$$

is similar to the first-order first-degree linear ordinary differential equation studied in the preceding section. [see (4.2)]. Clearly, when $n = 1$, it is a first-order - first-degree linear ordinary differential equation with constant coefficients. But it does not outwardly appear to be so when $n \neq 0$ or 1. Yet it turns out that even when n is different from 0 or 1, (4.19) can easily be reduced to linear form by a procedure first suggested by Bernouilli.

4.3.1 The Bernouilli Suggestion

Introduce a function $u_0(x)$.

$$u_0(x) = [u(x)]^{1-n} \quad . \tag{4.20}$$

Therefore,

$$u(x) = [u_0(x)]^{\left(\frac{1}{1-n}\right)} \ , \tag{4.21}$$

and

$$Du(x) = \left(\frac{1}{1-n}\right) \cdot [u_0(x)]^{\{(\frac{1}{1-n})-1\}} \cdot \frac{\mathrm{d}u_0(x)}{\mathrm{d}x} \quad . \tag{4.22}$$

Express $u(x)$ and $Du(x)$ as in (4.21) and represent (4.19) in terms of $u_0(x)$.

$$\left(\frac{1}{1-n}\right) \cdot [u_0(x)]^{\{(\frac{1}{1-n})-1\}} \cdot \frac{\mathrm{d}u_0(x)}{\mathrm{d}x} + M(x)\,[u_0(x)]^{(\frac{1}{1-n})} = N(x)\,[u_0(x)]^{(\frac{n}{1-n})} \quad . \tag{4.23}$$

Multiply both sides by $(1-n)\,[u_0(x)]^{-\{(\frac{1}{1-n})-1\}}$ to arrive at the following linear equation:

$$\begin{aligned} Du_0(x) + (1-n)M(x)u_0(x) &= (1-n)N(x)\,[u_0(x)]^0 \\ &= (1-n)N(x) \quad . \end{aligned} \tag{4.24}$$

Bernouilli Equation: Solution

One can make the Bernouilli equation—that is (4.24)—look very similar to (4.2) by introducing the notation

$$M_0(x) \;=\; (1-n)M(x) \;\;;\;\; N_0(x) \;=\; (1-n)N(x) \;\;, \tag{4.25}$$

and writing the resultant differential equation as

$$[D + M_0(x)]u_0(x) = N_0(x) \quad . \tag{4.26}$$

Recall that a similar looking differential equation (4.2), that is,

$$[D + M(x)]u(x) = N(x) \;\;, \tag{4.27}$$

has its solution embedded in (4.10) and (4.15). And both of these can immediately be transferred to relate to the newest version of the Bernouilli differential equation—that is (4.26)—merely by adding the subscript 0 to the functions $u(x)$, $M(x)$, and $N(x)$. So, according to (4.10) and (4.15), the solution to Bernouilli differential equation (4.26) is as follows:

$$u_0(x) = \frac{1}{W_0(x)} \left[\int W_0(x)\, N_0(x)\, \mathrm{d}x + \sigma_1\right] \;\;, \tag{4.28}$$

where, σ_1 is an arbitrary constant and

$$W_0(x) = \exp\left(\int M_0(x)\,\mathrm{d}x\right) \quad . \tag{4.29}$$

With $M_0(x)$ and $N_0(x)$ as given in (4.25), and $u_0(x)$ as in (4.21), (4.28), and (4.29) provide the desired solution of the differential equation (4.19).

4.3.2 *Examples Group II*

Make use of (4.20)–(4.29), and solve several Bernouilli equations of the form (4.19).

$$(1)\ \ Solve\ :\quad [D+2]\,u(x) \;=\; 3\;[u(x)]^{-1} \tag{4.30}$$

Equation (4.30): Solution

Here, $M(x) = 2$; $N(x) = 3$; and $n = -1$. Therefore, $(1-n) = 2$ and $(1-n)M(x) = M_0(x) = 4$, $(1-n)N(x) = N_0(x) = 6$, and the resultant differential equation, in the form of (4.26), is

$$[D+4]\,u_0(x) \;=\; 6 \quad . \tag{4.31}$$

According to (4.21), (4.28), and (4.29), we have

$$\begin{aligned} W_0(x) &= \exp\left(\int M_0(x)\,\mathrm{d}x\right) \\ &= \exp\left(\int 4\,\mathrm{d}x\right) \;=\; \exp(4x) \quad , \end{aligned} \tag{4.32}$$

and

$$\begin{aligned} u_0(x) &= \frac{1}{W_0(x)}\left[\int W_0(x)\,N_0(x)\,\mathrm{d}x + \sigma_1\right] \\ &= \exp(-4x)\left[\int 6\,\exp(4x)\,\mathrm{d}x + \sigma_1\right] \\ &= \frac{6}{4} + \sigma_1\,\exp(-4x) \;=\; [u(x)]^{1-n} \;=\; [u(x)]^2 \quad . \end{aligned} \tag{4.33}$$

Therefore, according to (4.33), the solution to the Bernouilli differential equation (4.30) is

$$u(x) = \pm\left[\frac{6}{4} + \sigma_1\,\exp(-4x)\right]^{\frac{1}{2}} \quad . \qquad :\ (1) \tag{4.34}$$

$$(2)\ \ Solve\ :\quad [D+x]\,u(x) \;=\; x\;[u(x)]^2 \tag{4.35}$$

Equation (4.35): Solution

Here, $M(x) = N(x) = x$ and $n = 2$. Therefore, $M_0(x) = N_0(x) = -x$, $(1-n) = -1$, and the resultant differential equation, in the form of (4.26), is

$$[D - x]\, u_0(x) = -x \tag{4.36}$$

According to (4.21), (4.28), and (4.29), one has

$$\begin{aligned} W_0(x) &= \exp\left(\int M_0(x)\, \mathrm{d}x\right) \\ &= \exp\left(\int -x\, \mathrm{d}x\right) = \exp\left(-\frac{x^2}{2}\right), \end{aligned} \tag{4.37}$$

and

$$\begin{aligned} u_0(x) &= \frac{1}{W_0(x)}\left[\int W_0(x)\, N_0(x)\, \mathrm{d}x + \sigma_1\right] \\ &= \exp\left(\frac{x^2}{2}\right)\left[\int -x\, \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x + \sigma_1\right] \\ &= 1 + \sigma_1\, \exp\left(\frac{x^2}{2}\right) = [u(x)]^{1-n} = [u(x)]^{-1} . \end{aligned} \tag{4.38}$$

Therefore, the solution to the Bernouilli differential equation (4.35) is

$$u(x) = \left[1 + \sigma_1\, \exp\left(\frac{x^2}{2}\right)\right]^{-1} . \qquad : \quad (2) \tag{4.39}$$

Next, solve

$$(3) \quad : \quad [D + x]\, u(x) = x\, [u(x)]^3 \tag{4.40}$$

Equation (4.40): Solution

Here, $M(x) = N(x) = x$ and $n = 3$. Therefore, $M_0(x) = N_0(x) = -2x$ and $(1-n) = -2$. The resultant differential equation, in the form of (4.26), is

$$[D - 2x]\, u_0(x) = -2x \tag{4.41}$$

As usual, one has

$$W_0(x) = \exp\left(\int -2x\, \mathrm{d}x\right) = \exp\left(-x^2\right), \tag{4.42}$$

and

$$u_0(x) = \exp\left(x^2\right)\left[\int (-2\,x)\,\exp\left(-x^2\right)\mathrm{d}x + \sigma_1\right]$$
$$= 1 + \sigma_1\,\exp\left(x^2\right) \;=\; [u(x)]^{1-n} \;=\; [u(x)]^{-2}\quad . \tag{4.43}$$

Therefore, the solution to the Bernouilli differential equation (4.35) is

$$u(x) = \pm\left[1 + \sigma_1\,\exp\left(x^2\right)\right]^{-\frac{1}{2}}\quad .\qquad :\;(3) \tag{4.44}$$

Next, work out

$$(4)\;:\quad [D + x]\,u(x) \;=\; \exp(x^2)\,[u(x)]^3 \tag{4.45}$$

Equation (4.45): Solution

Here, $M(x) = x$; $N(x) = \exp(x^2)$; and $n = 3$. Therefore, $(1-n) = -2$; $M_0(x) = -2\,x$; $N_0(x) = -2\,\exp(x^2)$. The resultant differential equation, in the form of (4.26), is

$$[D - 2\,x]\,u_0(x) \;=\; -2\,\exp(x^2) \tag{4.46}$$

Following the usual protocol, one gets

$$W_0(x) = \exp\left(\int -2\,x\,\mathrm{d}x\right) \;=\; \exp\left(-x^2\right)\;, \tag{4.47}$$

and

$$u_0(x) = \exp\left(x^2\right)\left[\int \exp\left(-x^2\right).\{-2\,\exp(x^2)\}\,\mathrm{d}x + \sigma_1\right]$$
$$= \exp(x^2)\,[-2\,x + \sigma_1] \;=\; [u(x)]^{1-n} \;=\; [u(x)]^{-2}\quad . \tag{4.48}$$

Therefore, the solution to the Bernouilli differential equation (4.45) is

$$u(x) = \pm\frac{1}{\sqrt{\exp(x^2)\,[-2\,x + \sigma_1]}}\quad .\qquad :\;(4) \tag{4.49}$$

Another equation that to solve is

$$(5)\;:\quad \left[D + x^2\right]u(x) \;=\; \exp(x^3)\,[u(x)]^4 \tag{4.50}$$

Equation (4.50): Solution

Here, $M(x) = x^2$; $N(x) = \exp(x^3)$; and $n = 4$. Therefore, $(1-n) = -3$; $M_0(x) = -3\,x^2$; $N_0(x) = -3\,\exp(x^3)$. The resultant differential equation, in the form of (4.26), is

$$\left[D - 3\,x^2\right]u_0(x) \;=\; -3\,\exp(x^3) \tag{4.51}$$

As usual, one can write

$$W_0(x) = \exp\left(\int -3\,x^2\,\mathrm{d}x\right) \;=\; \exp\left(-x^3\right)\;, \tag{4.52}$$

and

$$\begin{aligned} u_0(x) &= \exp\left(x^3\right)\left[\int \exp\left(-x^3\right)\,.\{-3\,\exp(x^3)\}\,\mathrm{d}x + \sigma_1\right] \\ &= \exp(x^3)\,[-3\,x + \sigma_1] \;=\; [u(x)]^{1-n} \;=\; [u(x)]^{-3}\;. \end{aligned} \tag{4.53}$$

Therefore, the solution to the Bernouilli differential equation (4.50) is $u(x)$ such that

$$[u(x)]^3 = \frac{1}{\exp(x^3)\,[-3\,x + \sigma_1]} \tag{4.54}$$

whose three solutions, according to standard rules of algebra, are

$$\begin{aligned} u(x) &= \frac{1}{\sqrt[3]{\exp(x^3)\,[-3\,x + \sigma_1]}} \\ u(x) &= \frac{(-1)^{\frac{2}{3}}}{\sqrt[3]{\exp(x^3)\,[-3\,x + \sigma_1]}} \\ u(x) &= \frac{(-1)^{\frac{4}{3}}}{\sqrt[3]{\exp(x^3)\,[-3\,x + \sigma_1]}}\;. \qquad :\; (5) \end{aligned} \tag{4.55}$$

The next few Bernouilli type equations that are solved are numbered (6)–(10) and are given below in (4.56).

$$\begin{aligned} (6)\;:\quad & [D + x]\,u(x) \;=\; x^3\;[u(x)]^{-3} \\ (7)\;:\quad & [D + (1/x)]\,u(x) \;=\; x\;[u(x)]^2 \\ (8)\;:\quad & [D + (1/x)]\,u(x) \;=\; x\;[u(x)]^3 \\ (9)\;:\quad & [D + (1/x)]\,u(x) \;=\; x\;[u(x)]^{-2} \\ (10)\;:\quad & \left[D + x^2\right]u(x) \;=\; \exp(x)\;[u(x)]^2\;. \end{aligned} \tag{4.56}$$

Equations (4.56): Solution

(6): Here, $M(x) = x$; $N(x) = x^3$; and $n = -3$. Therefore, $(1 - n) = 4$; $M_0(x) = 4\,x$; $N_0(x) = 4\,x^3$. The resultant differential equation, in the form of (4.26), is

$$[D + 4\,x]\,u_0(x) \;=\; 4\,x^3\;. \tag{4.57}$$

As usual, we can write

$$W_0(x) = \exp\left(\int 4\,x\,\mathrm{d}x\right) \;=\; \exp\left(2\,x^2\right)\;, \tag{4.58}$$

and

$$\begin{aligned} u_0(x) &= \exp\left(-2\,x^2\right)\,\left[\int \exp\left(2\,x^2\right)\,.\{4\,x^3\}\,\mathrm{d}x + \sigma_1\right] \\ &= \exp\left(-2\,x^2\right)\,\left[\frac{1}{2}\exp\left(2\,x^2\right)(2x^2-1) + \sigma_1\right]\;. \end{aligned} \tag{4.59}$$

Therefore, the solution to the Bernouilli differential equation (4.56)-(6) is $u(x)$ such that

$$[u(x)]^4 = \left[\frac{1}{2}(2x^2-1) + \sigma_1\exp\left(-2\,x^2\right)\right] \tag{4.60}$$

whose four solutions, according to standard rules of algebra, are

$$\begin{aligned} u(x) &= \pm\left[\frac{1}{2}(2x^2-1) + \sigma_1\exp\left(-2\,x^2\right)\right]^{\frac{1}{4}} \\ u(x) &= \pm(i)\left[\frac{1}{2}(2x^2-1) + \sigma_1\exp\left(-2\,x^2\right)\right]^{\frac{1}{4}}\;. \end{aligned} \qquad :\;(6) \tag{4.61}$$

(7): Here, $M(x) = 1/x$; $N(x) = x$; and $n = 2$. Therefore, $(1-n) = -1$; $M_0(x) = -1/x$; $N_0(x) = -x$. Thus,

$$[D - (1/x)]\,u_0(x) \;=\; -\,x\,, \tag{4.62}$$

$$W_0(x) = \exp\left(\int -\frac{\mathrm{d}x}{x}\right) \;=\; \exp\left(-\log x\right) \;=\; \frac{1}{x}\,, \tag{4.63}$$

and

$$\begin{aligned} u_0(x) &= x\,\left[\int (-\mathrm{d}x) + \sigma_1\right] \\ &= [u(x)]^{-1}\;. \end{aligned} \tag{4.64}$$

Therefore, the solution to the Bernouilli differential equation (4.56)-(7) is

$$u(x) = \frac{1}{x\,[-x+\sigma_1]}\;. \tag{4.65}$$

(8): Here, $M(x) = 1/x$; $N(x) = x$; and $n = 3$. Therefore, $(1-n) = -2$; $M_0(x) = -2/x$; $N_0(x) = -2\,x$, and

$$[D - (2/x)]\,u_0(x) = -2\,x\,, \tag{4.66}$$

$$W_0(x) = \exp\left(\int -2\,\frac{\mathrm{d}x}{x}\right) = \exp\left(-2\log x\right) = \frac{1}{x^2}\,, \tag{4.67}$$

$$\begin{aligned} u_0(x) &= x^2\left[-2\int\frac{\mathrm{d}x}{x} + \sigma_1\right] \\ &= [u(x)]^{-2}\ . \end{aligned} \tag{4.68}$$

Accordingly, the solution to the Bernouilli differential equation (4.56)-(8) is

$$u(x) = \pm\left[x^2\,\{-2\,\log(x) + \sigma_1\}\right]^{-\frac{1}{2}}\ . \tag{4.69}$$

(9): Here, $M(x) = 1/x$; $N(x) = x$; and $n = -2$. Therefore, $(1-n) = 3$; $M_0(x) = 3/x$; $N_0(x) = 3\,x$. Thus,

$$[D + (3/x)]\,u_0(x) = 3\,x\,, \tag{4.70}$$

$$W_0(x) = \exp\left(\int 3\,\frac{\mathrm{d}x}{x}\right) = \exp\left(3\log x\right) = x^3\,, \tag{4.71}$$

and

$$\begin{aligned} u_0(x) &= x^{-3}\left[3\int x^4\mathrm{d}x + \sigma_1\right] \\ &= x^{-3}\left[\left(\frac{3}{5}\right)x^5 + \sigma_1\right] = [u(x)]^3\ . \end{aligned} \tag{4.72}$$

Therefore, there are three solutions to the Bernouilli differential equation (4.56)-(9): namely

$$\begin{aligned} u(x) &= \left[\frac{\left(\frac{3}{5}\right)x^5 + \sigma_1}{x^3}\right]^{\frac{1}{3}}\ , \\ u(x) &= (-1)^{\frac{2}{3}}\left[\frac{\left(\frac{3}{5}\right)x^5 + \sigma_1}{x^3}\right]^{\frac{1}{3}}\ , \\ u(x) &= (-1)^{\frac{4}{3}}\left[\frac{\left(\frac{3}{5}\right)x^5 + \sigma_1}{x^3}\right]^{\frac{1}{3}}\ . \qquad :\ (9) \end{aligned} \tag{4.73}$$

(10): Here, $M(x) = x^2$; $N(x) = \exp(x)$; and $n = 2$. Therefore, $(1-n) = -1$; $M_0(x) = -x^2$; $N_0(x) = -\exp(x)$. Thus,

$$\left[D - x^2\right] u_0(x) = -\exp(x)\,, \tag{4.74}$$

$$W_0(x) = \exp\left(\int -x^2\,\mathrm{d}x\right) = \exp\left(-\frac{x^3}{3}\right), \tag{4.75}$$

and

$$u_0(x) = \exp\left(\frac{x^3}{3}\right)\left[-\int \exp\left(x - \frac{x^3}{3}\right)\mathrm{d}x + \sigma_1\right] . \tag{4.76}$$

The solution to the Bernouilli differential equation (4.56)-(10) is

$$u(x) = [u_0(x)]^{-1} . \tag{4.77}$$

4.3.3 Problems Group II

Given below are a set of ten Bernouilli differential equations similar to (4.19). The relevant choices for n, $M(x)$ and $N(x)$, are as noted in (4.78)-(1)–(4.78)-(10). [Hint: Compare (4.19)–(4.77).]

$$\begin{aligned}
&(1): && M(x) = 3\ , && N(x) = 2\ , && n = -1\ .\\
&(2): && M(x) = 2x\ , && N(x) = 3x\ , && n = 2\ .\\
&(3): && M(x) = 3x\ , && N(x) = 2x\ , && n = 3\ .\\
&(4): && M(x) = x\ , && N(x) = 3\exp(x^2)\ , && n = 3\ .\\
&(5): && M(x) = x^2\ , && N(x) = 6\exp(x^3)\ , && n = 4\ .\\
&(6): && M(x) = x\ , && N(x) = 2x^3\ , && n = -3\ .\\
&(7): && M(x) = \frac{1}{x}\ , && N(x) = 3x\ , && n = 2\ .\\
&(8): && M(x) = \frac{2}{x}\ , && N(X) = 2x\ , && n = 2\ .\\
&(9): && M(x) = \frac{2}{x}\ , && N(X) = 2x\ , && n = 3\ .\\
&(10): && M(x) = \frac{2}{x}\ , && N(X) = 3x\ , && n = -2\ .
\end{aligned} \tag{4.78}$$

Chapter 5
Green's Function Laplace Transforms



5.1 Green's Function

Consider an inhomogeneous linear ordinary differential equation.

$$V(x)\,Y(x) = F(x)\ . \tag{5.1}$$

The differential operator $V(x)$, the solution $Y(x)$, and quite possibly also the inhomogeneous term represented by the arbitrary function $F(x)$, involve constants and derivatives with respect to the variable x. The objective of the present exercise is to determine the solution, $Y(x)$, of the differential equation (5.1) with specified boundary conditions.

In the preceding chapters, various straightforward methods for solving homogeneous linear ordinary differential equations were discussed. For inhomogeneous linear ordinary differential equations, those methods required more effort. Especially, the particular integral, I_{pi}, needs to be worked out ab initio for every different inhomogeneous term $F(x)$. But there exists another powerful methodology named after George *Green*[26.] whereby Green's function, once and for all, helps solve the differential equation for arbitrary values of the inhomogeneous term. Therefore, for a particular differential operator, in addition to the homogeneous solution, only Green's function needs to be worked out.

The Green function procedure is well-suited to studying inhomogeneous ordinary differential equations. A given Green's function always refers to a particular differential operator. For the present case, $V(x)$ is the relevant differential operator. While there is no total agreement on this issue, most of the available literature, for instance, Dean J. *Duffy*[24.], would accept the following definition of Green's function, $G(x, x')$, that is, relevant to the differential operator $V(x)$.

$$V(x)\,G(x, x') = \delta(x - x')\ . \tag{5.2}$$

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Here, $\delta(x - x')$ is Dirac's delta function. Note, some discussion of the *Dirac*[33] delta function is provided in the Appendix.

If Green's function, $G(x, x')$, as defined in (5.2), can be worked out, the desired solution of the differential equation $V(x)\,Y(x) = F(x)$ can be represented as (5.3) given below.

$$Y(x) = \int F(x')\,G(x, x')\,\mathrm{d}x' \,. \tag{5.3}$$

The correctness of this assertion is assured if it can be shown to be consistent with the fundamental equation (5.1). To check this fact, let us proceed as follows. Multiply (5.3) from the left by $V(x)$.

$$\begin{aligned} V(x)\,Y(x) &= V(x) \int F(x')\,G(x, x')\,\mathrm{d}x' \\ &= \int F(x')\;V(x)\;G(x, x')\,\mathrm{d}x' \;. \end{aligned} \tag{5.4}$$

And use (5.2).

$$\begin{aligned} V(x)\,Y(x) &= V(x) \int F(x')\,G(x, x')\,\mathrm{d}x' \\ &= \int F(x')V(x)\,G(x, x')\,\mathrm{d}x' = \int F(x')\,\delta(x - x')\,\mathrm{d}x' = F(x)\,. \end{aligned} \tag{5.5}$$

Notice that (5.5) properly reproduced (5.1).

Let us work with boundary conditions that apply at the ends of the allowed physical space. Assume that such physical space ranges over the interval $0 \le x \le \pi$—and choose the following as the boundary conditions:

$$Y(x = 0) \equiv Y(0) = 0 \quad , \quad Y(x = \pi) \equiv Y(\pi) = 0\,. \tag{5.6}$$

Because (5.3) holds for arbitrary $F(x')$, these boundary conditions,

$$\begin{aligned} Y(0) &= \int_0^\pi F(x')\,G(0, x')\,\mathrm{d}x' = 0\,, \\ Y(\pi) &= \int_0^\pi F(x')\,G(\pi, x')\,\mathrm{d}x' = 0\,, \end{aligned} \tag{5.7}$$

demand

$$G(0, x') = 0 \quad , \quad G(\pi, x') = 0\,. \tag{5.8}$$

An important property of $G(x, x')$ is that (5.3) applies to any inhomogeneous term $F(x')$ subject only to the requirement that $F(x')$ is well behaved. Clearly,

if one can determine Green's function, the relevant inhomogeneous differential equation (5.1) is readily solved and (5.3) provides the desired solution $Y(x)$ for arbitrary choice of the inhomogeneous term $F(x')$. The question then is to learn how to travel the route that leads to the needed Green's function.

There are several possible routes: some more tortuous than others. And while the reached Green's function may behave as expected, the chances are that at least occasionally it would lead to solutions that are too complicated to be useful.

5.2 Solving Differential Equations

5.2.1 Eigenfunction Expansion

Most differential equations have two components: One that deals with the homogeneous part of the equation and the other that involves the inhomogeneous part. Fortunately, the various techniques for handling the homogeneous part are similar to those discussed in detail in Chaps. 3 and 4 . Therefore, it can be assumed that the solution to the homogeneous part can be worked out.

In this subsection, we show that a Green's function may be calculated by using eigenfunctions. That is, information obtained from solving the homogeneous part of the differential equation. Indeed, a Green's function is often fathered by the homogeneous part of the referencing differential equation. And once Green's function is calculated, it can be used to solve the inhomogeneous part of the referencing differential equation. And that can be done for arbitrary choices of the inhomogeneous term.

Let us consider the following linear ordinary differential equation.

$$\left[D^2 + \alpha^2\right] y_n(x) = \gamma_n y_n(x) \,, \quad n \,=\, 0,\, 1\,,\, 2\,,\, 3\,, \ldots \tag{5.9}$$

This equation is similar to differential equations we have learned to solve in Chaps. 3, and 4. Equation (5.9) is actually an eigenvalue *equation*[34]. The Hamiltonian, $\left[D^2 + \alpha^2\right]$, is Hermitian and the eigenfunctions, $y_n(x)$,

$$y_n(x) \;=\; A \sin\left[n\,x\right] + B \cos\left[n\,x\right] \;, \tag{5.10}$$

are orthogonal and complete. Also the eigenvalues, γ_n, are real. As shown below, for (5.9) both the eigenvalues γ_n and the eigenfunction $y_n(x)$ are readily evaluated. The eigenfunction can also be made orthonormal so that the relevant integral of eigenfunctions $y_{n_1}(x)$ and $y_{n_2}(x)$, calculated over the prescribed spatial range, equals $\delta_{n_1,\,n_2}$.

$$\int_0^{\pi} y_{n_1}(x)\, y_{n_2}(x)\, \mathrm{d}x \;=\; \delta_{n_1,\,n_2}. \tag{5.11}$$

We require that the eigenfunctions $y_n(x)$ remain valid in the physical interval $0 \le x \le \pi$ and obey the boundary conditions

$$y_n(0) \;=\; 0 \quad , \quad y_n(\pi) = 0 \quad . \tag{5.12}$$

The first boundary condition, namely $y_n(x = 0) = 0$, is satisfied only by the sine term in (5.10). And because at $x = 0$ the cosine term is non-vanishing, cosines must be excluded altogether from (5.10). Similarly, regarding the sign term, the second boundary condition—namely $y_n(x = \pi) = 0$—requires that for other than the trivial case $n = 0$, $sin\,[n\,\pi]$ is equal to zero only when n is an integer. Therefore,

$$y_n(x) \;=\; |\rho_n|\, sin(n\,x) \quad , \quad for \;\; n \;=\; 1,\; 2,\; 3,\; \ldots \tag{5.13}$$

Without loss of generality, one can use only the positive n. The normalization requirement,

$$\int_0^{\pi} [y_n(x)]\,[y_n(x)]^*\,\mathrm{d}x \;=\; 1 \;= |\rho_n|^2 \int_0^{\pi} sin(n\,x)^2\,\mathrm{d}x \;=\; |\rho_n|^2\,\frac{\pi}{2}\;, \tag{5.14}$$

sets the constant

$$|\rho_n| \;=\; \sqrt{\frac{2}{\pi}} \tag{5.15}$$

and the eigenfunctions $y_n(x)$.

$$y_n(x) \;=\; \sqrt{\frac{2}{\pi}}\, sin(n\,x)\;. \tag{5.16}$$

Using (5.9) and (5.16), we can write

$$\begin{aligned} [D^2 + \alpha^2]\, y_n(x) \;&=\; \gamma_n\, y_n(x) \\ = [D^2 + \alpha^2] \sqrt{\frac{2}{\pi}}\, sin(n\,x) \;&=\; [\alpha^2 - n^2] \sqrt{\frac{2}{\pi}}\, sin(n\,x) \;=\; [\alpha^2 - n^2]\, y_n(x)\;. \end{aligned} \tag{5.17}$$

Equation (5.17) provides the relevant eigenvalues γ_n.

$$\gamma_n = (\alpha^2 - n^2). \tag{5.18}$$

5.2.2 *Green's Function Calculated*

As stated above, the eigenfunctions $y_n(x)$ are orthonormal and complete. [Note : For observational convenience, (5.1) is reprinted below as (5.19).] Therefore, when $Y(x)$

$$F(x) = V(x)\,Y(x) \tag{5.19}$$

is a function in the same domain as the eigenfunctions $y_n(x)$ and obeys the same boundary conditions, it may be represented as a linear combination of the stated eigenfunctions.

$$Y(x) = \sum_{n=0}^{\infty} b_n\, y_n(x). \tag{5.20}$$

To study this matter, further proceed as follows:

Set $V(x) = [D^2 + \alpha^2]$. Reprint (5.2). And insert the $Y(x)$, given in above (5.20), into (5.19).

$$\begin{aligned} V(x) &= [D^2 + \alpha^2]\ ; \\ V(x)\,G(x, x') &= \delta(x - x')\ ; \\ F(x) = [D^2 + \alpha^2] \sum_{n=0}^{\infty} b_n\, y_n(x) \ &= \ \sum_{n=0}^{\infty} b_n\, [D^2 + \alpha^2]\, y_n(x)\ . \end{aligned} \tag{5.21}$$

Use the first row of (5.17) and thereby replace $[D^2 + \alpha^2]\, y_n(x)$ by $\gamma_n\, y_n(x)$. As a result, (5.21) becomes

$$F(x) = \sum_{n=0}^{\infty} b_n\, \gamma_n\, y_n(x)\ \ . \tag{5.22}$$

The constant b_n needs to be determined. To that end, multiply (5.22) by $y_m(x)$ and integrate over the relevant space.

$$\begin{aligned} \int_0^{\pi} y_m(x)\, F(x)\, \mathrm{d}x &= \int_0^{\pi} \sum_{n=0}^{\infty} b_n\, \gamma_n\, y_m(x)\, y_n(x) \mathrm{d}x \\ = \sum_{n=0}^{\infty} b_n\, \gamma_n \int_0^{\pi} y_m(x)\, y_n(x) \mathrm{d}x &= \sum_{n=0}^{\infty} b_n\, \gamma_n\, \delta_{n,m} \ = \ b_m\, \gamma_m\ . \end{aligned} \tag{5.23}$$

Now divide both sides of (5.23) by γ_m.

$$\frac{1}{\gamma_m}\int_0^{\pi} y_m(x)\,F(x)\,\mathrm{d}x = b_m\,. \tag{5.24}$$

For convenience, change the variables from m to n and x to x' in (5.24). This gives us the desired b_n.

$$b_n = \frac{1}{\gamma_n}\int_0^{\pi} y_n(x')\,F(x')\,\mathrm{d}x'\,. \tag{5.25}$$

Combine the result in (5.25) for b_n with (5.20) to get

$$\begin{aligned} Y(x) = \sum_{n=0}^{\infty} b_n.\,y_n(x) \;&=\; \sum_{n=1}^{\infty}\frac{1}{\gamma_n}\int_0^{\pi} y_n(x')\,F(x')\,\mathrm{d}x'.\,y_n(x) \\ &= \int_0^{\pi}\left[\sum_{n=0}^{\infty}\frac{1}{\gamma_n}\,y_n(x)\,y_n(x')\right]F(x')\,\mathrm{d}x'\,. \end{aligned} \tag{5.26}$$

But according to (5.3), we have

$$Y(x) = \int_0^{\pi}\left[G(x,x')\right]F(x')\,\mathrm{d}x'\,. \tag{5.27}$$

The eigenfunction Green's function $G(x,x')$ is now readily identified by treating simultaneously (5.26) and (5.27). We get

$$G(x,x') \;=\; \sum_{n=0}^{\infty}\frac{1}{\gamma_n}\,y_n(x)\,y_n(x') \;=\; \sum_{n=0}^{\infty}\frac{1}{(\alpha^2-n^2)}\,y_n(x)\,y_n(x'). \tag{5.28}$$

[Note: The eigenvalue γ_n used in (5.28) is picked up from (5.18).]

It should be emphasized that the denominator $(\alpha^2 - n^2)$ in (5.28) is particular to the differential operator $\left[D^2+\alpha^2\right]$. And according to (5.27), for any arbitrary choice of $F(x')$, Green's function (5.28) should provide the desired solution $Y(x)$.

In other words, the function $Y(x)$ in (5.29),

$$\left[D^2+\alpha^2\right]Y(x) \;=\; F(x)\;, \tag{5.29}$$

is determined by the relationship

$$\begin{aligned} Y(x) &= \sum_{n=0}^{\infty}\frac{1}{(\alpha^2-n^2)}\,y_n(x)\int_0^{\pi} y_n(x')\,F(x')\,\mathrm{d}x' \\ &= \left(\frac{2}{\pi}\right)\sum_{n=0}^{\infty}\frac{1}{(\alpha^2-n^2)}\,sin(n\,x)\int_0^{\pi} sin(n\,x')\,F(x')\,\mathrm{d}x'. \end{aligned} \tag{5.30}$$

5.2.3 Examples Group I

Use (5.16) and (5.30) and work out solutions to the differential equation $\left(D^2 + \alpha^2\right) Y(x) = F(x)$ for the following choices of $F(x)$.

$$\begin{aligned} F(x) &= sin(x) \ . && (1) \\ F(x) &= sin(x)\, cos(x) \ . && (2) \\ F(x) &= x \ . && (3) \end{aligned} \tag{5.31}$$

5.2.4 Solution

(1)

Insert $F(x') = sin(x')$ in (5.30).

$$Y(x) \ = \ \left(\frac{2}{\pi}\right) \sum_{n=0}^{\infty} \frac{1}{(\alpha^2 - n^2)}\, sin(n\,x) \int_0^{\pi} sin(n\,x')\, sin(x')\, \mathrm{d}x'$$

Because the integral $\int_0^{\pi} sin(n\,x')\, sin(x')\, \mathrm{d}x' = (\frac{\pi}{2})\delta(n-1)$, only the $n = 1$ term survives. Thus,

$$(1) \ : \ Y(x) = \sum_{n=0}^{\infty} \left(\frac{2}{\pi}\right) \frac{1}{(\alpha^2 - n^2)}\, sin(x) \left(\frac{\pi}{2}\right) \delta(n-1) = \frac{sin(x)}{\alpha^2 - 1}. \tag{5.32}$$

(2)

Insert $F(x') = sin(x')cos(x') = \frac{sin(2\,x')}{2}$ in (5.30).

$$Y(x) \ = \ \left(\frac{1}{2}\right)\left(\frac{2}{\pi}\right) \sum_{n=0}^{\infty} \frac{1}{(\alpha^2 - n^2)}\, sin(n\,x) \int_0^{\pi} sin(n\,x')\, sin(2\,x')\, \mathrm{d}x'$$

Because the integral $\int_0^{\pi} sin(n\,x')\, sin(2\,x')\, \mathrm{d}x' = (\frac{\pi}{2})\delta(n-2)$, only the $n = 2$ term survives. Thus,

$$(2) \ : \ Y(x) = \sum_{n=0}^{\infty} \left(\frac{1}{\pi}\right) \frac{1}{(\alpha^2 - n^2)}\, sin(n\,x) \left(\frac{\pi}{2}\right) \delta(n-2) = \left(\frac{1}{2}\right) \frac{sin(2x)}{\alpha^2 - 4}. \tag{5.33}$$

(3)

Insert $F(x') = x'$ in (5.30).

$$: \ Y(x) \ = \ \left(\frac{2}{\pi}\right) \sum_{n=0}^{\infty} \frac{1}{(\alpha^2 - n^2)}\, sin(n\,x) \int_0^{\pi} sin(n\,x')\, x'\, \mathrm{d}x' \tag{5.34}$$

Do the integral $\int_0^\pi \, sin(n\,x')\,x'\,\mathrm{d}x'$ by parts.

$$\int_0^\pi \, sin(n\,x)\,x\,\mathrm{d}x \;=\; \left[-\frac{x\,cos(n\,x)}{n}\right]_0^\pi + \frac{1}{n}\int_0^\pi \, cos(n\,x)\mathrm{d}x$$
$$= -\frac{\pi\,cos(n\,\pi)}{n} + \frac{1}{n}\left[\frac{sin(nx)}{n}\right]_0^\pi \;=\; \frac{\pi(-1)^{n+1}}{n} + 0 \;. \qquad (5.35)$$

In order to calculate $Y(x)$, the result in (5.35) above should be inserted into (5.34).

$$(3) \;:\; Y(x) = \sum_{n=0}^{\infty} 2\, sin(n\,x)\,\frac{(-1)^{n+1}}{n(\alpha^2 - n^2)} \;. \qquad (5.36)$$

Note: The result $Y(x)$ given in (5.36) looks more complicated than one would have anticipated. Therefore, it is important to check its accuracy. To that end let us plug this $Y(x)$ into (5.29) and check whether it actually leads to $F(x) = x$. That is, let us work with equations

$$\left[D^2 + \alpha^2\right] Y(x) \;=\; F(x)$$
$$= \left[D^2 + \alpha^2\right] \sum_{n=0}^{\infty} 2\, sin(n\,x)\,\frac{(-1)^{n+1}}{n(\alpha^2 - n^2)} \;=\; F(x) \;=\; x$$
$$= \sum_{n=0}^{\infty} 2\,(\alpha^2 - n^2)\, sin(n\,x)\,\frac{(-1)^{n+1}}{n(\alpha^2 - n^2)} \;=\; \sum_{n=0}^{\infty} 2\,\frac{(-1)^{n+1}}{n}\, sin(n\,x) \;. \qquad (5.37)$$

Again one wonders whether the right-hand side of (5.37) is indeed equal to x. Actually, it looks much like a Fourier series. If so, is it the Fourier expansion of x? Let us check.

Fourier expansion of x.

$$x \;=\; a_0 + \sum_{n=1}^{\infty} a_n\, cos(n\,x) + \sum_{n=1}^{\infty} b_n\, sin(nx) \;, \qquad (5.38)$$

where

$$a_0 \;=\; \frac{1}{2\,\pi}\int_{-\pi}^{\pi} x\,\mathrm{d}x \;=\; 0 \;,\quad a_n \;=\; \frac{1}{\pi}\int_{-\pi}^{\pi} x\, cos(nx)\mathrm{d}x \;=\; 0 \quad ,$$
$$b_n \;=\; \frac{1}{\pi}\int_{-\pi}^{\pi} x\, sin(nx)\mathrm{d}x \;=\; \frac{1}{\pi}\left[\frac{-x\,cos(n\,x)}{n}\right]_{-\pi}^{\pi} + \frac{1}{\pi}\left[\frac{sin(n\,x)}{n}\right]_{-\pi}^{\pi}$$
$$= \frac{1}{\pi}\left[\frac{2\,\pi\,(-1)^{n+1}}{n}\right] \;=\; 2\,\frac{(-1)^{n+1}}{n} \;. \qquad (5.39)$$

Using b_n from (5.39), (5.38) gives

$$x \;=\; \sum_{n=0}^{\infty} 2\,\frac{(-1)^{n+1}}{n}\,sin(n\,x) \;\;. \tag{5.40}$$

Comparison of (5.37) and (5.40) confirms the result $F(x) = x$. Q.E.D.

Results for Differential Operator $V(x) = \left(D^2 - \frac{9}{4}\right)$

Set a numerical value for the variable α.

$$\alpha = \frac{3}{2}\,. \tag{5.41}$$

As a result, the previously calculated expression for the eigenfunction Green's function given in (5.28), i.e.,

$$G(x, x') \;=_{eigenfunction\ result}\; \sum_{n=0}^{\infty} \frac{1}{(\alpha^2 - n^2)}\,y_n(x)\,y_n(x') \;\;,$$

will change. And we will have the following eigenfunction Green's function when $\alpha = \frac{3}{2}$.

$$G(x, x') \;=_{\,eigenfunction\ result}\; \sum_{n=0}^{\infty} \frac{4}{(9 - 4\,n^2)}\,y_n(x)\,y_n(x'). \tag{5.42}$$

5.3 Calculation by Approaching Delta Function

As noticed above, the eigenfunctions Green's function leads to infinite series with results that are often both difficult to work out and complicated to work with. On many occasions, a better approach is to work directly with the defining equation of Green's function that involves Dirac's delta function. And then approach the delta function singularity from either sides, thereby obtaining a closed-form expression for Green's function. To demonstrate this procedure and to compare with the previous results, we work with a similar differential equation to that used for deriving the eigenfunctions Green's function.

Consider the differential operator $V(x) = \left[D^2 + \frac{9}{4}\right]$ and its Green's function $G(x, x_0)$. [Note: Compare (5.2)]

$$\left[D^2 + \frac{9}{4}\right]\,G(x, x_0) = \delta(x - x_0) \;\;. \tag{5.43}$$

As before, x and x_0 are chosen to lie within the interval $0 \leq x \leq \pi$. The delta function in (5.43) refers to the physical separation $(x - x_0)$. There are two possibilities for reaching the singularity at $x = x_0$. First: x can approach x_0 from below. Second: x approaches x_0 from above. For either of these options, as long x dos not touch x_0, the delta function $\delta(x - x_0)$ itself is vanishing and (5.43) reduces to the following:

$$\left[D^2 + \frac{9}{4}\right] G(x, x_0) \; = \; 0 \; , \tag{5.44}$$

with the solution

$$G(x, x_0) \; = \; A \, sin\left(\frac{3}{2}x\right) + B \, cos\left(\frac{3}{2}x\right) \quad . \tag{5.45}$$

Let us examine how the constants A and B are affected as the position x moves within the intervals $x_0 > x \geq 0$ and $\pi \geq x > x_0$.

One of the relevant boundary conditions (5.8), namely

$$G(x = 0, x_0) \; = \; 0 \; , \tag{5.46}$$

excludes the cosines in the region $x_0 > x \geq 0$. Therefore, we have

$$G(x, x_0) = A \, sin\left(\frac{3}{2}x\right) \quad , \quad x_0 > x \geq 0. \tag{5.47}$$

In contrast, for the region $\pi \geq x > x_0$ we need to have, $G(x = \pi, x_0) \; = \; 0$. Therefore, the sine term is excluded. As such, we can write

$$G(x, x_0) = B \, cos\left(\frac{3}{2}x\right) \quad , \quad \pi \geq x > x_0. \tag{5.48}$$

Our next task is to calculate the constants A and B. To that end, we can impose the continuity requirement at x_0 so the result is the same whether x_0 is approached from below or from above. We get

$$A \, sin\left(\frac{3}{2}x_0\right) \; = \; B \, cos\left(\frac{3}{2}x_0\right) \quad . \tag{5.49}$$

Clearly, we need one more relationship to determine A and B, that is, provided by continuity of the differential $\left[\frac{dG(x,x_0)}{dx}\right]_{x=x_0}$. This is best calculated by integrating (5.43) in the following manner

$$lim_{\epsilon\to o}\left\{\int_{x_0-\epsilon}^{x_0+\epsilon}\left[D^2+\frac{9}{4}\right]G(x,x_0)\,\mathrm{d}x\right\} = lim_{\epsilon\to o}\left\{\int_{x_0-\epsilon}^{x_0+\epsilon}\delta(x-x_0)\right\}$$
$$= lim_{\epsilon\to o}\left[\frac{\mathrm{d}G}{\mathrm{d}x}\right]_{x_0-\epsilon}^{x_0+\epsilon} + \left(\frac{9}{4}\right) lim_{\epsilon\to o}\left\{\int_{x_0-\epsilon}^{x_0+\epsilon} G(x,x')\mathrm{d}x\right\} = 1\,. \tag{5.50}$$

Equation (5.50) leads to

$$lim_{\epsilon\to o}\left\{\left[\frac{\mathrm{d}G}{\mathrm{d}x}\right]_{x_0+\epsilon} - \left[\frac{\mathrm{d}G}{\mathrm{d}x}\right]_{x_0-\epsilon}\right\} + 0 = 1\,. \tag{5.51}$$

Using the expressions (5.48) and (5.47), (5.51) is rewritten as

$$-B\left(\frac{3}{2}\right) sin\left(\frac{3}{2}x_0\right) - A\left(\frac{3}{2}\right) cos\left(\frac{3}{2}x_0\right) = 1\,. \tag{5.52}$$

Solving (5.49) and (5.52) together gives

$$A = -\left(\frac{2}{3}\right) cos\left(\frac{3}{2}x_0\right)\;;\; B = -\left(\frac{2}{3}\right) sin\left(\frac{3}{2}x_0\right)\,. \tag{5.53}$$

Inserting A and B given in (5.53) into (5.47) and (5.48) leads to the desired Green's function.

$$G(x,x_0) = -\left(\frac{2}{3}\right) cos\left(\frac{3}{2}x_0\right) sin\left(\frac{3}{2}x\right)\;,\quad x_0 > x \geq 0. \tag{5.54}$$

$$G(x,x_0) = -\left(\frac{2}{3}\right) sin\left(\frac{3}{2}x_0\right) cos\left(\frac{3}{2}x\right)\;,\quad \pi \geq x > x_0. \tag{5.55}$$

5.3.1 Examples Group II

Use the closed-form Green's function procedure outlined in (5.54) and (5.55) and work out the differential equation (5.29) for the same two choices of $F(x)$ given in (5.31) that were treated with the eigenfunctions Green's function $\left(D^2+\alpha^2\right)Y(x) = F(x)$.

$$F(x) = sin(x)\;.\;(1)$$
$$F(x) = sin(x)\,cos(x)\;.\;(2)\;. \tag{5.56}$$

5.3.2 Solution

F(x) = sin(x) : (1)

Rewrite (5.27).

$$Y(x) = \int_0^{\pi} G(x, x_0)\, F(x_0)\, \mathrm{d}x_0 \, .$$

Exchange variables x and x_o.

$$Y(x_0) = \int_0^{\pi} G(x_0, x)\, F(x)\, \mathrm{d}x \, .$$

Because the current Green's function is produced by a self-adjoint differential operator, $\left[D^2 + \alpha^2\right]$, its Green's function is symmetric, i.e., $G(x_0, x) = G(x, x_0)$. Therefore, we can write the above as

$$Y(x_0) = \int_0^{\pi} G(x, x_0)\, F(x)\, \mathrm{d}x \, . \tag{5.57}$$

Insert the $G(x, x_0)$ given in (5.54) and (5.55) into (5.57). For $F(x) = sin(x)$, we get

$$\begin{aligned} Y(x_0) &= -\left(\frac{2}{3}\right) cos\left(\frac{3}{2}x_0\right) \left[\int_0^{x_0} sin\left(\frac{3}{2}x\right) sin(x)\, \mathrm{d}x\right] \\ &\quad - \left(\frac{2}{3}\right) sin\left(\frac{3}{2}x_0\right) \left[\int_{x_0}^{\pi} cos\left(\frac{3}{2}x\right) sin(x)\, \mathrm{d}x\right] \\ &= -\left(\frac{2}{3}\right) cos\left(\frac{3}{2}x_0\right) \left[sin\left(\frac{x_0}{2}\right) - \left(\frac{1}{5}\right) sin\left(\frac{5}{2}x_0\right)\right] \\ &\quad - \left(\frac{2}{3}\right) sin\left(\frac{3}{2}x_0\right) \left[-cos\left(\frac{x_0}{2}\right) + \left(\frac{1}{5}\right) cos\left(\frac{5}{2}x_0\right)\right] \, . \end{aligned} \tag{5.58}$$

Equation (5.58) can be further organized.

$$\begin{aligned} Y(x_0) &= -\left(\frac{2}{3}\right) \left\{\sin\left(\frac{x_0}{2}\right) cos\left(\frac{3}{2}x_0\right) - cos\left(\frac{x_0}{2}\right) sin\left(\frac{3}{2}x_0\right)\right\} \\ &\quad + \left(\frac{2}{15}\right) \left[sin\left(\frac{5}{2}x_0\right) cos\left(\frac{3}{2}x_0\right) - cos\left(\frac{5}{2}x_0\right) sin\left(\frac{3}{2}x_0\right)\right] \\ &= -\left(\frac{2}{3}\right) sin\left[\frac{x_0}{2} - 3\frac{x_0}{2}\right] + \left(\frac{2}{15}\right) sin\left[\frac{5\,x_0}{2} - \frac{3\,x_0}{2}\right] \\ &= (2/3)\ sin(x_0) + (2/15)\ sin(x_0) = (4/5)\ sin(x_0) \, . \end{aligned} \tag{5.59}$$

It is straightforward to check the corresponding prediction of the eigenfunctions Green's function. For that purpose, we go to (5.32) and record its result obtained from summing an infinite series. That is

$$(1) \;:\; Y(x) = \frac{sin(x)}{\alpha^2 - 1} \; . \tag{5.60}$$

The only change necessary in the eigenfunctions result given in (5.60) is to change α^2 to $\frac{9}{4}$. Then, we would have $Y(x) = \frac{4\,sin(x)}{5}$, which is exactly the same as the current result given above in (5.59). Q.E.D.

F(x) = sin(x) cos(x) : (2)

In order to calculate $Y(x_0)$ with $F(x) = sin(x)\,cos(x)$, we need to insert Green's function provided in (5.54) and (5.55) into (5.57). We get

$$\begin{aligned} Y(x_0) = &-\left(\frac{1}{3}\right) cos\left(\frac{3}{2}x_0\right)\left[\int_0^{x_0} sin\left(\frac{3}{2}x\right) sin(2\,x)\,\mathrm{d}x\right] \\ &-\left(\frac{1}{3}\right) sin\left(\frac{3}{2}x_0\right)\left[\int_{x_0}^{\pi} cos\left(\frac{3}{2}x\right) sin(2\,x)\,\mathrm{d}x\right] \; . \end{aligned} \tag{5.61}$$

After doing the integrals, the above can be written as

$$\begin{aligned} Y(x_0) &= -\left(\frac{1}{3}\right) cos\left(\frac{3}{2}x_0\right)\left[sin\left(\frac{x_0}{2}\right) - \left(\frac{1}{7}\right) sin\left(\frac{7\,x_0}{2}\right)\right] \\ &\quad -\left(\frac{1}{3}\right) sin\left(\frac{3}{2}x_0\right)\left[cos\left(\frac{x_0}{2}\right) + \left(\frac{1}{7}\right) cos\left(\frac{7\,x_0}{2}\right)\right] \\ &= -\frac{1}{3}\left[sin\left(\frac{x_0}{2}\right) cos\left(\frac{3\,x_0}{2}\right) + \sin\left(\frac{3\,x_0}{2}\right) cos\left(\frac{x_0}{2}\right)\right] \\ &\quad + \frac{1}{21}\left[sin\left(\frac{7\,x_0}{2}\right) cos\left(\frac{3\,x_0}{2}\right) - sin\left(\frac{3\,x_0}{2}\right) cos\left(\frac{7\,x_0}{2}\right)\right] \\ &= -\left(\frac{1}{3}\right) sin\left(\frac{x_0}{2} + \frac{3\,x_0}{2}\right) + \left(\frac{1}{21}\right) sin\left(\frac{7\,x_0}{2} - \frac{3\,x_0}{2}\right) \\ &= -\left(\frac{1}{3}\right) sin\,(2\,x_0) + \left(\frac{1}{21}\right) sin\,(2\,x_0) \\ &= -\left(\frac{2}{7}\right) sin(2\,x_0) \; . \end{aligned} \tag{5.62}$$

Again, it is straightforward to check the corresponding prediction of the eigenfunctions Green's function. For that purpose, we go to (5.33) and record its result obtained from summing an infinite series. That is,

$$(2) \;:\; Y(x) = \left(\frac{1}{2}\right) \frac{sin(2x)}{\alpha^2 - 4} \; . \tag{5.63}$$

The only change necessary in the eigenfunctions result given in (5.63) is to change α^2 to $\frac{9}{4}$. Then, we would have $Y(x) = -\left(\frac{2}{7}\right) sin(2\,x_0)$, which is exactly the current result given above in (5.62). Q.E.D.

5.3.3 Examples Group III

Work out Green's function relevant to differential equation (5.64).

$$\left[\frac{d}{dt} - \alpha\right] x(t) \;=\; t^2 \, \exp(\omega\, t) \;\; . \tag{5.64}$$

5.3.4 Solution

Similar to (5.1), (5.2), (5.3) we can write

$$V(t)\, Y(t) = F(t) \qquad , \tag{5.65}$$

$$V(t)\, G(t, t_0) = \delta(t - t_0) \;\; , \tag{5.66}$$

$$Y(t) = \int_0^{\pi} F(t_0)\, G(t, t_0)\, \mathrm{d}t_0 \;\; . \tag{5.67}$$

Next, we choose

$$V(t) \;=\; \left[\frac{\mathrm{d}}{\mathrm{d}t} - \alpha\right] \;\; , \;\; Y(t) \;=\; x(t) \;\; , \;\; F(t) \;=\; t^2 \, \exp(\omega\, t) \;\; , \tag{5.68}$$

work out the relevant Green's function, $G(t, t_0)$, determine the solution $x(t)$ of the differential equation (5.69) within the domain $\{\pi \geq t > 0\}$ and obeying the boundary condition $\{\lim_{t=0+\epsilon} x(t) = 0\;.\}$

$$\left[\frac{\mathrm{d}}{\mathrm{d}t} - \alpha\right] x(t) \;=\; t^2 \, \exp(\omega\, t) \;\; . \tag{5.69}$$

In order to follow the instructions given above, begin with (5.66).

$$\left[\frac{\mathrm{d}}{\mathrm{d}t} - \alpha\right] G(t, t_0) \;=\; \delta(t - t_0) \;\; . \tag{5.70}$$

For $t > t_0$, the delta function $\delta(t - t_0)$ is vanishing. Then, the truncated version of differential equation (5.70) is easy to solve and we get

$$G(t, t_0) = \exp\left[\alpha\,(t - t_0)\right] \;\; , \quad \textit{for} \;\; t \;>\; t_0 \; . \tag{5.71}$$

Therefore, according to (5.67) and (5.68) the solution $x(t)$ is

$$x(t) \equiv Y(t) \;=\; \int_0^{\pi} F(t_0)\, G(t, t_0)\, \mathrm{d}t_0 \;\; . \tag{5.72}$$

Set $F(t_0) = t_0^2\, \exp(\omega\, t_0)$. Insert it in (5.72). Use (5.71) and rewrite the result as (5.73).

$$\begin{aligned}
x(t) &= \int_0^{\pi} t_0^2\, \exp(\omega\, t_0)\, G(t, t_0)\, \mathrm{d}t_0 \;=\; \int_0^{t} t_0^2\, \exp(\omega\, t_0)\, \exp\left[\alpha\,(t - t_0)\right]\, \mathrm{d}t_0 \\
&= \exp\left(\alpha\, t\right) \int_0^{t} t_0^2\, \exp\left[t_0\;(\omega - \alpha)\right]\, \mathrm{d}t_0 \\
&= \exp\left(\alpha\, t\right) \left[\left\{\frac{t_0^2\, \exp\left[t_0\,(\omega - \alpha)\right]}{(\omega - \alpha)}\right\}_0^{t}\right] - \left(\frac{2}{\omega - \alpha}\right) \int_0^{t} t_0\, \exp\left[t_0\;(\omega - \alpha)\right]\, \mathrm{d}t_0 \\
&= \frac{t^2\, \exp\omega\, t}{\omega - \alpha} - \left(\frac{2\, \exp\alpha\, t}{\omega - \alpha}\right) \left\{\frac{t_0\, \exp\left[t_0\,(\omega - \alpha)\right]}{(\omega - \alpha)}\right\}_0^{t} \\
&+ \frac{2\, \exp\alpha\, t}{(\omega - \alpha)^2} \int_0^{t} \exp\left[t_0\;(\omega - \alpha)\right]\, \mathrm{d}t_0 \\
&= (\exp\omega\, t) \left[\frac{t^2}{(\omega - \alpha)} - \frac{2\, t}{(\omega - \alpha)^2} + \frac{2}{(\omega - \alpha)^3}\right] - \frac{2\, \exp\alpha\, t}{(\omega - \alpha)^3} .
\end{aligned} \tag{5.73}$$

In order to ascertain whether the $x(t)$ given in (5.73) satisfies the differential equation (5.69), we proceed as follows.

$$\begin{aligned}
\frac{\mathrm{d}x(t)}{\mathrm{d}t} - \alpha\, x \;&=\; (\exp\omega\, t) \left[\frac{2\, t}{(\omega - \alpha)} - \frac{2}{(\omega - \alpha)^2}\right] - \frac{2\, \alpha\, \exp\alpha\, t}{(\omega - \alpha)^3} \\
&+ \omega\; (\exp\omega\, t) \left[\frac{t^2}{(\omega - \alpha)} - \frac{2\, t}{(\omega - \alpha)^2} + \frac{2}{(\omega - \alpha)^3}\right] \\
&- \alpha\; (\exp\omega\, t) \left[\frac{t^2}{(\omega - \alpha)} - \frac{2\, t}{(\omega - \alpha)^2} + \frac{2}{(\omega - \alpha)^3}\right] + \frac{2\, \alpha\, \exp\alpha\, t}{(\omega - \alpha)^3} \; .
\end{aligned} \tag{5.74}$$

Equation (5.74) meanders and is quite long. We write it in a more organized form below.

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} - \alpha\, x(t) \;=\; t^2\, \exp(\omega\, t) \left[\frac{\omega - \alpha}{\omega - \alpha}\right] + \exp(\alpha\, t) \left[\frac{2\alpha - 2\,\alpha}{(\omega - \alpha)^3}\right]$$
$$+\, \exp(\omega\, t) \left[\frac{(2\,\omega - 2\,\alpha)}{(\omega - \alpha)^3} - \frac{2}{(\omega - \alpha)^2}\right] \;=\; t^2\, \exp(\omega\, t)\;. \tag{5.75}$$

This is indeed the result expected from the differential equation (5.69). Therefore, the correctness of the $x(t)$ given in (5.73) is verified. Q.E.D.

5.4 Laplace Transform

It is helpful to use a general notation for Laplace transforms. Consider some function j of t, as in $j(t)$, where j is any lowercase character. Denote its Laplace transform by the same uppercase character J. But signify it as a function of s, as in $J(s)$. As such the inverse transform of $J(s)$ is $j(t)$. These statements are displayed below in (5.76).

$$\Lambda_s\{j(t)\} = lim_{t_0 -> \infty} \left[\int_0^{t_0} j(t)\, \exp(-s\, t)\, \mathrm{d}t\right] \equiv J(s),$$
$$\Lambda_s^{-1}\{J(s)\} \equiv j(t)\;. \tag{5.76}$$

For a function $j(t)$ such that $j(t) = 0$ when $t < 0$, the Laplace integral $\Lambda_s\{j(t)\}$ specifies the Laplace transform $J(s)$.

$$J(s) \;\equiv\; \Lambda_s\{j(t)\} = \int_0^{\infty} j(t)\, \exp(-s\, t)\, \mathrm{d}t. \tag{5.77}$$

$J(s)$ exists for $s > 0$ provided $j(t)$ satisfies the following conditions:

(1) $j(t)$=0 for $t < 0$
(2) $j(t)$ is continuous, or at least piecewise continuous, in every interval
(3) $t^n f(t) < \infty$ as $t- > 0$ for some number n, where $n < 1$.

5.4.1 Table

$f(t)$	$F(s) = lim_{t_0->\infty}\left[\int_0^{t_0} f(t)\exp(-s\,t)\,\mathrm{d}t\right]$
(1). 1	$1/s$
(2). $a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(s) + a_2 F_2(s)$
(3). $H(t-c)$	$\exp(-c\,s)/s$
(4). $H(t-c)f(t-c)$	$\exp(c\,s)f(s)$
(5). t	$1/s^2$
(6). t^2	$2!/s^3$
(7). t^3	$3!/s^4$
(8). $t^n\ ,\ n = 1, 2, 3$	$n!/s^{n+1}$
(9). $t^p\ , p > -1$	$\frac{G(p+1)}{s^{p+1}}$
(10). $\exp(a\,t)\ ,\ s > a$	$1/(s-a)$
(11). $t\exp(a\,t)\ ,\ s > a$	$1/(s-a)^2$
(12). $\delta(t-a)$	$\exp(-a\,s)$
(13). $t^n\exp(a\,t)\ ,\ s > a$	$n!/(s-a)^{n+1}$
(14). $cos(b\,t)$	$s/(s^2+b^2)$
(15). $\exp(a\,t)cos(b\,t)$	$(s-a)/[(s-a)^2+b^2]$
(16). $cosh(b\,t)$	$s/(s^2-b^2)$
(17). $sin(b\,t)$	$b/(s^2+b^2)$
(18). $\exp(a\,t)sin(b\,t)$	$b/[(s-a)^2+b^2]$
(19). $sinh(b\,t)$	$b/(s^2-b^2)$
(20). $t\,sin(b\,t)$	$(2\,a\,s)/(s^2+b^2)^2$
(21). $t\,cos(b\,t)$	$(s^2-b^2)/(s^2+b^2)^2$
(22). $t\,sinh(a\,t)$	$2\,a\,s/(s^2-a^2)^2$
(23). $t\,cosh(a\,t)$	$s^2+a^2(s^2-a^2)^2$
(24). $sin(a\,t) - a\,t\,cos(a\,t)$	$2\,a^3/(s^2+a^2)^2$
(25). $sin(a\,t) + a\,t\,cos(a\,t)$	$2\,a\,s^2/(s^2+a^2)^2$
(26). $cos(a\,t) - a\,t\,sin(a\,t)$	$s\,(s^2-a^2)/(s^2+a^2)^2$
(27). $cos(a\,t) + a\,t\,sin(a\,t)$	$s\,(s^2+3\,a^2)/(s^2+a^2)^2$
(28). $sin(a\,t+b)$	$[s\,sin(b) + a\,cos(b)]/(s^2+a^2)$
(29). $cos(a\,t+b)$	$[s\,cos(b) - a\,sin(b)]/(s^2+a^2)$
(30). $\frac{1}{t}f(t)$	$\int_t^\infty F(u)\,\mathrm{d}u$
(31). $f(c\,t)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$
(32). $\int_0^t f(v)\,\mathrm{d}v$	$\frac{1}{s}\,F(s)$
(33). $\int_0^t f(t-\tau)\,g(\tau)\,\mathrm{d}\tau$	$F(s)\,G(s)$
(34). $cos(a\,t+b)$	$[s\,cos(b) - a\,sin(b)]/(s^2+a^2)$
(35). $f'(t)$	$s\,F(s) - f(0)$
(36). $f'''(t)$	$s^3F(s) - s^2f(0) - sf'(0) - f''(0)$
(37). $\frac{[\exp(a\,t)-\exp(b\,t)]}{(a-b)}$	$(s-a)^{-1}(s-b)^{-1}$
(38). $erfc\left(\frac{a}{2\sqrt{t}}\right)$	$\frac{1}{s}\exp\left(-a\sqrt{s}\right)$
(39). $\frac{1}{\sqrt{\pi\,t}}\exp\left(\frac{-a^2}{4\,t}\right)$	$\frac{1}{\sqrt{s}}\exp(-a\,\sqrt{s})$
(40). $\frac{a}{2\sqrt{\pi\,t^3}}\exp\left(\frac{-a^2}{4\,t}\right)$	$\exp(-a\,\sqrt{s})$
(41). $\frac{a\exp(a\,t)-b\exp(b\,t)}{a-b}$	$s(s-a)^{-1}(s-b)^{-1}$
(42). $\sqrt{t}$	$\sqrt{\frac{\pi}{4\,s}}$

(5.78)

5.5 Computation

Laplace transform of a function $f(t)$—which is at least piecewise continuous on a specified interval—is defined through the use of an integral and is informatively denoted as $\Lambda_s\{f(t)\}$. In other words,

$$\Lambda_s\{f(t)\} \;=\; lim_{t_0->\infty}\left[\int_0^{t_0} f(t)\exp(-s\,t)\,\mathrm{d}t\right] \;=\; F(s). \tag{5.79}$$

To get a feel for how the integration process (5.79) actually unfolds, let us work out a few simple examples.

(1) : For $f(t) = t^n$, compute $\Lambda_s\{f(t)\} \;=\; F(s)$.
To solve (1) write:

$$\begin{aligned}
\Lambda_s\{t^n\} &= lim_{t_0->\infty}\int_0^{t_0} t^n\,\exp(-s\,t)\,\mathrm{d}t \equiv \left(\frac{n}{s}\right)\int_0^{\infty} t^{(n-1)}\,\exp(-s\,t)\,\mathrm{d}t \\
&= \left(\frac{n(n-1)}{s^2}\right)\int_0^{\infty} t^{(n-2)}\,\exp(-s\,t)\,\mathrm{d}t \\
&= \left(\frac{n(n-1)(n-2)}{s^3}\right)\int_0^{\infty} t^{(n-3)}\,\exp(-s\,t)\,\mathrm{d}t \\
&= \ldots \\
\textit{Therefore}\;,\;\; \Lambda_s\{t^n\} &= \frac{n!}{s^{(n+1)}}\;.
\end{aligned} \tag{5.80}$$

In particular, using $n = 0$, and $n = 1$, we get $\Lambda_s\{1\} = \frac{1}{s}$, and $\Lambda_s\{t\} = \frac{1}{s^2}$.

(2) : For $f(t) = sin(b\,t)$, compute $\Lambda_s\{f(t)\} \;=\; F(s)$.
To solve (2) write:

$$\begin{aligned}
f(t) &= sin(b\,t)\;,\;\; for\;\; t > 0\;, \\
F(s) &= lim_{t_0->\infty}\left[\int_0^{t_0} sin(b\,t)\exp(-s\,t)\,\mathrm{d}t\right]\;,\;\; s > 0\;,
\end{aligned}$$

Doing integration by parts gives:

$$\begin{aligned}
&= lim_{t_0->\infty}\left[-\left\{\frac{cos(b\,t)\;\exp(-s\,t)}{b}\right\}_0^{t_0} - \frac{s}{b}\int_0^{t_0} cos(b\,t)\exp(-s\,t)\,\mathrm{d}t\right]\;, \\
&= \frac{1}{b} - \left(\frac{s^2}{b^2}\right) lim_{t_0->\infty}\left[\int_0^{t_0} sin(b\,t)\exp(-s\,t)\,\mathrm{d}t\right], \\
F(s) &= \frac{1}{b} - \left(\frac{s^2}{b^2}\right)F(s)\;.
\end{aligned} \tag{5.81}$$

Therefore,

$$\Lambda_s\{sin(b\,t)\} = F(s) = \left(\frac{b}{s^2+b^2}\right) \quad . \tag{5.82}$$

NOTE: For convenience, instead of the proper form, i.e., $lim_{t_0->\infty} \int_0^{t_0}$, of the infinite integrals that occur here, heretofore we shall use only their improper form, i.e., $\int_0^{\infty}$.

(3) : For $f(t) = c_1 f_1(t) + c_2 f_2(t)$ compute $\Lambda_s\{f(t)\}$.
To solve (3) write:

$$\begin{aligned}\Lambda_s\{c_1 f_1(t) + c_2 f_2(t)\} &= \int_0^{\infty} f(t)\exp(-s\,t)\,\mathrm{d}t \\ &= c_1 \int_0^{\infty} f_1(t)\exp(-s\,t)\,\mathrm{d}t \;+\; c_2 \int_0^{\infty} f_2(t)\exp(-s\,t)\,\mathrm{d}t \\ &= c_1\,\Lambda_s\{f_1(t)\} + c_2\,\Lambda_s\{f_2(t)\} \\ &= c_1\,F_1(s) + c_2\,F_2(s). \end{aligned} \tag{5.83}$$

(4) : For $f(t) = \exp(at)$, compute $\Lambda_s\{f(t)\} = F(s)$.
To solve (4) write:

$$\begin{aligned}\Lambda_s\{\exp(at)\} &= \int_0^{\infty} \exp[-(s-a)\,t]\,\mathrm{d}t\ , \\ &= \left(\frac{1}{s-a}\right) \quad , \quad \textit{only for } s > a\ . \end{aligned} \tag{5.84}$$

Here, it is convenient to briefly describe $H(t-a)$, the Heaviside step function. However, for greater detail see (5.92). The step function $H(t-a)$ is defined for $a \geq 0$ by the relationship

$$\begin{aligned} H(t-a) &= 1, \quad \textit{for } \; t > a \\ &= 0, \quad \textit{for } \; t < a\ . \end{aligned} \tag{5.85}$$

(5) : For $f(t) = f_1(t-a)\,H(t-a)$, compute $\Lambda_s\{f_1(t-a)\,H(t-a)\} = F(s)$.

To solve (5) write:

$$\begin{aligned}\Lambda_s\{f_1(t-a)\,H(t-a)\} &= \int_0^{\infty} f_1(t-a)\,H(t-a)\ \exp(-s\,t)\mathrm{d}t \\ &= \int_a^{\infty} f_1(t-a)\ \exp(-s\,t)\,\mathrm{d}t\ . \end{aligned}$$

Set $t - a = y$: then $\mathrm{d}t = \mathrm{d}y$ and the above becomes

$$\begin{aligned}\Lambda_s\{f_1(t-a)\,H(t-a)\} &= \int_0^\infty f_1(y)\exp[-s\,(y+a)]\,\mathrm{d}y \\ &= \exp(-s\,a)\int_0^\infty f_1(t)\exp[-s\,t]\,\mathrm{d}t \\ &= \exp(-s\,a)\;\Lambda_s\{f_1(t)\}\;. \end{aligned} \tag{5.86}$$

(6) : Consider $f(t) = \delta(t-a)$. Note that $a \geq 0$.

To solve (6) write:

$$\begin{aligned}\Lambda_s\{\delta(t-a)\} &= \int_0^\infty \exp(-s\,t)\delta(t-a)\,\mathrm{d}t \\ &= \exp(-s\,a)\;. \end{aligned} \tag{5.87}$$

(7) : First consider $f(t) = \frac{\mathrm{d}f_i(t)}{\mathrm{d}t}, f(t) = \frac{\mathrm{d}^2 f_i(t)}{\mathrm{d}t^@}, f(t) = \frac{\mathrm{d}^3 f_i(t)}{\mathrm{d}t^3}$.

To solve (7) write:

$$\begin{aligned}\Lambda_s\left[\frac{\mathrm{d}^3 f_i(t)}{\mathrm{d}t^3}\right] &= \int_0^\infty \exp(-s\,t)\frac{\mathrm{d}^3 f_i(t)}{\mathrm{d}t^3}\,\mathrm{d}t \\ &= \left(\exp(-s\,t)\,\frac{\mathrm{d}^2 f_i(t)}{\mathrm{d}t^2}\right)|_0^\infty + s\int_0^\infty \exp(-s\,t)\frac{\mathrm{d}^2 f_i(t)}{\mathrm{d}t^2}\,\mathrm{d}t. \\ &= -\left(\frac{\mathrm{d}^2 f_i(t)}{\mathrm{d}t^2}\right)_{t=0} + s\left[\left(\exp(-s\,t)\,\frac{\mathrm{d}f_i(t)}{\mathrm{d}t}\right)|_0^\infty + s\int_0^\infty \exp(-s\,t)\frac{\mathrm{d}f_i(t)}{\mathrm{d}t}\,\mathrm{d}t\right] \\ &= -\left(\frac{\mathrm{d}^2 f_i(t)}{\mathrm{d}t^2}\right)_{t=0} - s\left(\frac{\mathrm{d}f_i(t)}{\mathrm{d}t}\right)_{t=0} + s^2\int_0^\infty \exp(-s\,t)\frac{\mathrm{d}f_i(t)}{\mathrm{d}t}\,\mathrm{d}t \\ &= -\left(\frac{\mathrm{d}^2 f_i(t)}{\mathrm{d}t^2}\right)_{t=0} - s\left(\frac{\mathrm{d}f_i(t)}{\mathrm{d}t}\right)_{t=0} + s^2\,\left[\exp(-s\,t)f_i(t)\right]|_0^\infty + s^3\,\Lambda_s[f_i(t)] \\ &= -s^0\left(\frac{\mathrm{d}^2 f_i(t)}{\mathrm{d}t^2}\right)_{t=0} - s\left(\frac{\mathrm{d}f_i(t)}{\mathrm{d}t}\right)_{t=0} - s^2 f_i(t)_{t=0} + s^3\,\Lambda_s[f_i(t)]. \end{aligned}$$

Next proceed by induction and write $\Lambda_s\left[\frac{\mathrm{d}^n f_i(t)}{\mathrm{d}t^n}\right]$ *as follows* :

$$\begin{aligned}\Lambda_s\left[\frac{\mathrm{d}^n f_i(t)}{\mathrm{d}t^n}\right] &= -s^0\left(\frac{\mathrm{d}^{n-1} f_i(t)}{\mathrm{d}t^2}\right)_{t=0} - s\left(\frac{\mathrm{d}^{n-2} f_i(t)}{\mathrm{d}t^{n-2}}\right)_{t=0} - s^2\left(\frac{\mathrm{d}^{n-3} f_i(t)}{\mathrm{d}t^{n-3}}\right)_{t=0} \\ &\quad - \ldots - s^{n-2}\left(\frac{\mathrm{d}f_i(t)}{\mathrm{d}t}\right)_{t=0} - s^{n-1} f_i(t)_{t=0} + s^n\,\Lambda_s[f_i(t)] \end{aligned} \tag{5.88}$$

(8) : Consider $f(t) = \int_0^t f_i(x)\mathrm{d}x$.
To solve (8) write:

$$\Lambda_s\left(\int_0^t f_i(x)\mathrm{d}x\right) \;=\; \int_0^\infty \exp(-s\,t)\left[\int_0^t f_i(x)\mathrm{d}x\right]\mathrm{d}t\;,$$

$$= \left[\frac{\exp(-s\,t)}{-s}\int_0^t f_i(x)\mathrm{d}x\right]_{t=0}^{t=\infty} + \frac{1}{s}\int_0^\infty \exp(-s\,t) f_i(t)\mathrm{d}t$$

$$= 0 + \frac{\Lambda_s\left[f_i(t)\right]}{s} \;=\; \left(\frac{F_i(s)}{s}\right). \tag{5.89}$$

(9) : Consider $f(t) = cos(a\,t)$. It is convenient to use the representation

$$f(t) \;=\; cos(a\,t) = \frac{\exp(I\,a\,t) + \exp(-I\,a\,t)}{2}\;,$$

and work with $s > a$.

$$\Lambda_s[cos(a\,t)] \;=\; \frac{1}{2}\int_0^\infty \exp[(I\,a - s)\,t)]\,\mathrm{d}t + \frac{1}{2}\int_0^\infty \exp[(-I\,a - s)\,t)]\,\mathrm{d}t\;,$$

$$= \frac{1}{2}\left[\frac{\exp[(I\,a - s)\,t]}{I\,a - s}\right]_0^\infty + \frac{1}{2}\left[\frac{\exp[(-I\,a - s)\,t]}{-I\,a - s}\right]_0^\infty$$

$$= \frac{1}{2}\frac{-1}{I\,a - s} + \frac{1}{2}\frac{-1}{-I\,a - s} \;=\; \frac{s}{s^2 + a^2}. \tag{5.90}$$

(10) : Consider $f(t) = sin(a\,t)$. It is convenient to use the representation

$$sin(a\,t) = \frac{\exp(a\,t) - \exp(-a\,t)}{2I}$$

Work with $s > a$ and write:

$$\Lambda_s[sin(a\,t)] \;=\; \frac{1}{2\,I}\left[\int_0^t \exp[(a - s)\,t)]\,\mathrm{d}t - \int_0^t \exp[(-a - s)\,t)]\,\mathrm{d}t\right]\;,$$

$$= \frac{1}{2I}\left[\frac{\exp[(a - s)\,t]}{a - s}\right]_0^\infty - \frac{1}{2I}\left[\frac{\exp[(-a - s)\,t]}{-a - s}\right]_0^\infty$$

$$= \frac{1}{2I}\frac{-1}{a - s} - \frac{1}{2I}\frac{-1}{-a - s} \;=\; (-I)\frac{a}{s^2 - a^2}. \tag{5.91}$$

5.6 Heaviside Step Function

A slightly more detailed description of $H(t-a)$ is given below. The Heaviside step function (HSF) is defined as follows for $a \geq 0$.

$$\begin{aligned} H(t-a) \equiv u(t-a) \equiv u_a(t) &= 1 \ for \ t \geq a \\ &= 0 \ for \ t < a \quad . \end{aligned} \tag{5.92}$$

5.6.1 On- and Off Switches

The (HSF) is useful for representing switches that turn on and off at specific times. For instance, consider the following two problems.

Using the notation of (5.1) in the form

$$V(t)\, Y(t) = F(t)$$

define various switches by choosing their inhomogeneous term $F(t)$.

(HSF)-1 : First, if one is using a very simple switch that is on for $t > 0$ with a value of W_0 but turns off at $t \geq 5$ then one has

$$\begin{aligned} F(t) &= W_0, \quad if \ \ t < 5 \\ &= 0 \ , \quad if \ \ t \geq 5 \ . \end{aligned} \tag{5.93}$$

In terms of the Heaviside function, this process is displayed as

$$F(t) = W_0 \ \{1 - H(t-5)\} \quad . \tag{5.94}$$

(HSF)-2 : On the other hand, a more sophisticated switch is one that is:
on for $t < 5$ with value W_1;
goes off during $10 > t \geq 5$;
comes back on at $t = 10$ with value W_2
and stays on until t = 15.
Turns off at t = 15
and comes back on at t = 20 with strength W_3
and stays on until t = 25.
But at t = 25 the switch instantly adjusts to strength W_4
and stays on at that strength.
Thus, one works with a switch that operates with function $F(t)$ such that

$$\begin{aligned} F(t) &= W_1, \quad if\ t < 5 \\ &= 0\ , \quad if\ 5 \le t < 10 \\ &= W_2, \quad if\ 10 \le t < 15 \\ &= 0\ , \quad if\ 15 \le t < 20 \\ &= W_3, \quad if\ 20 \le t < 25 \\ &= W_4, \quad if\ t \ge 25\ . \end{aligned} \tag{5.95}$$

In terms of the Heaviside function, this process can be displayed as follows:

$$\begin{aligned} F(t) = W_1\ \{1\ -\ H(t-5)\}\ +\ W_2\ \{H(t-10)\ -\ H(t-15)\} \\ +\ W_3\ \{H(t-20) - H(t-25)\}\ +\ W_4\ \{H(t-25)\}\ . \end{aligned} \tag{5.96}$$

5.7 Solving Initial Value Problems

The process:

One works out the Laplace transform of both sides of the given differential equation. The initial conditions are inserted into the Laplace transformed equation. Generally, this simplifies the output variable a little bit. Next, one reorganizes the output variable by partial fraction decomposition. Then, one inverse Laplace transforms the resultant output, if needs be, by using Laplace transform inversion tables. This process will become clear as we work out several problems.

5.8 First-Order Differential Equations

Solution: (1)

Let us solve the following first-order differential equation with boundary condition $x(t) = 0$ for $t < 0$.

$$(1)\ : 2\frac{\mathrm{d}x(t)}{\mathrm{d}t} + 5\,x(t) = t\,\exp(-t)\ . \tag{5.97}$$

Solution of (1).

Laplace transform of both sides of differential equation (5.97)—namely equation (1)—are

$$2\ \{s\,X(s) - x(0)\} + 5\,X(s) = \frac{1}{(s+1)^2}\ . \tag{5.98}$$

A more organized form of (5.98) is

$$X(s) = \left(\frac{1}{2s+5}\right)\frac{1}{(s+1)^2} + \frac{2x(0)}{(2s+5)}\,. \tag{5.99}$$

To satisfy the initial boundary condition, set $x(0) = 0$.

$$X(s) = \left(\frac{1}{2s+5}\right)\frac{1}{(s+1)^2}\,. \tag{5.100}$$

Notice how Laplace transform converts a function of some variable—say, t—into a function of another variable—say, s.

5.8.1 *Partial Fraction Decomposition of (5.100)*

To decompose (5.100) into partial fractions express $X(s)$ as follows:

$$X(s) \equiv \frac{A}{(2s+5)} + \frac{B}{(s+1)} + \frac{C}{(s+1)^2} \tag{5.101}$$

Multiply the right-hand sides of (5.100) and (5.101)
by $(2s+5)(s+1)^2$ and equate the two results. [Note the objective here is to get unity on the left-hand side of the following (5.102)]. We get

$$\begin{aligned} 1 &= A(s+1)^2 + B(s+1)(2s+5) + C(2s+5) \\ &= s^2(A+2B) + s(2A+2C+7B) + (A+5C+5B). \end{aligned} \tag{5.102}$$

For (5.102) to hold for arbitrary values of s, terms with any particular power of s must be equal on both sides of this equation. Accordingly, comparison of the s^2, s, and s^0 terms gives

$$\begin{aligned} A + 2B &= 0\ ; \\ 2A + 2C + 7B &= 0\ ; \\ A + 5C + 5B &= 1\ . \end{aligned} \tag{5.103}$$

Equation (5.103) is readily solved and the result is

$$A = \frac{4}{9}\ ;\ B = -\frac{2}{9}\ ;\ C = \frac{1}{3}\,. \tag{5.104}$$

Accordingly, using (5.101) and (5.104), $X(s)$ is rewritten as

$$\begin{aligned} X(s) &= \frac{4}{9}\,\frac{1}{(2s+5)} - \frac{2}{9}\,\frac{1}{(s+1)} + \frac{1}{3}\,\frac{1}{(s+1)^2} \\ &= \frac{4}{18}\,\frac{1}{(s+5/2)} - \frac{2}{9}\,\frac{1}{(s+1)} + \frac{1}{3}\,\frac{1}{(s+1)^2}\ . \end{aligned} \tag{5.105}$$

Finally, by using the inverse Laplace transform tables, one inverts the above from $X(s)$ to $x(t)$. The result is

$$x(t) = \frac{4}{18}\,\exp\left(-\frac{5\,t}{2}\right) - \frac{2}{9}\,\exp(-t) + \frac{t}{3}\,\exp(-t)\ . \tag{5.106}$$

Checking Accuracy of the Result (5.106)

In order to check $x(t)$, as per (5.106), one uses the homogeneous part of the parent differential equation (5.97), namely $2\,\frac{\mathrm{d}x(t)}{\mathrm{d}t} + 5\,x(t)$ and checks to see whether it equals the inhomogeneous part, i.e., $t\,\exp(-t)$.

First, we calculate $\frac{\mathrm{d}x(t)}{\mathrm{d}t}$.

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -\frac{10}{18}\,\exp\left(-\frac{5\,t}{2}\right) + \frac{5}{9}\,\exp(-t) - \frac{t}{3}\,\exp(-t)\,, \tag{5.107}$$

Next, we calculate $2\,\frac{\mathrm{d}x(t)}{\mathrm{d}t} + 5\,x(t)$ and find it is equal to $t\,\exp(-t)$, as it must if the solution is correct.

$$\begin{aligned} 2\,\frac{\mathrm{d}x(t)}{\mathrm{d}t} + 5\,x(t) &= -\frac{10}{9}\,\exp\left(-\frac{5\,t}{2}\right) + \frac{10}{9}\,\exp(-t) - \frac{2\,t}{3}\,\exp(-t) \\ &\quad + \frac{20}{18}\,\exp\left(-\frac{5\,t}{2}\right) - \frac{10}{9}\,\exp(-t) + \frac{5\,t}{3}\,\exp(-t) \\ &= t\,\exp(-t)\ .\ Q\,E\,D\,. \end{aligned} \tag{5.108}$$

Solution: (2)

Again, with the same initial boundary condition, namely $x(t) = 0$ for $t < 0$, let us solve another first-order differential equation: namely (5.109).

$$(2)\ :\ 3\frac{\mathrm{d}x(t)}{\mathrm{d}t} - 7\,x(t) = cosh(a\,t). \tag{5.109}$$

Solution of (2).

Laplace transform of both sides of the differential equation (5.109)—namely equation (2)— are

$$3\ \{s\,X(s) - x(0)\} - 7\,X(s) = \frac{s}{(s^2 - a^2)}\ . \tag{5.110}$$

Or

$$X(s) = \frac{1}{(s^2 - a^2)}\left(\frac{s}{3\,s - 7}\right) + \frac{3\,x(0)}{(3\,s - 7)}\ . \tag{5.111}$$

To satisfy the initial boundary condition, set $x(0) = 0$.

$$X(s) = \frac{1}{(s - a)}\,\frac{1}{(s + a)}\left(\frac{s}{3\,s - 7}\right)\ . \tag{5.112}$$

Notice how Laplace transform converts a function of some variable—say, t—into a function of another variable—say, s.

5.8.2 *Partial Fraction Decomposition of (5.112)*

To decompose (5.112) into partial fractions, express $X(s)$ as follows:

$$X(s) \equiv \frac{A}{(s - a)} + \frac{B}{(s + a)} + \frac{C}{(3s - 7)} \tag{5.113}$$

Multiply the right-hand sides of (5.112) and (5.113)
by $(s - a)\,(s + a)\,(3\,s - 7)$ and equate the two results. [Note the objective here is to get s on the left-hand side.] We get

$$\begin{aligned} s &= A\,(s + a)\,(3\,s - 7) + B\,(s - a)\,(3\,s - 7) + C\,(s - a)(s + a) \\ &= s^2\,(3\,A + 3\,B + C) + s\,(3\,a\,A - 7\,A - 7\,B - 3\,a\,B) \\ &+ (-7\,a\,A + 7\,a\,B - a^2\,C)\ . \end{aligned} \tag{5.114}$$

If (5.114) is to hold for arbitrary values of s, terms with any particular power of s must be equal on both sides of this equation. Accordingly, comparison of the s^2, s, and s^0 terms gives

$$\begin{aligned} 3\,A + 3\,B + C &= 0\ ; \\ (3\,a\,A - 7\,A - 7\,B - 3\,a\,B) &= 1; \\ (-7\,a\,A + 7\,a\,B - a^2\,C) &= 0. \end{aligned} \tag{5.115}$$

Equation (5.115) is readily solved and the result is

$$A = \frac{3\,a + 7}{18\,a^2 - 98}\ ;\ B = -\frac{3\,a - 7}{18\,a^2 - 98}\ ;\ C = -\frac{42}{18\,a^2 - 98} \tag{5.116}$$

Accordingly, using (5.113) and (5.116), one calculates $X(s)$.

$$X(s) = \frac{1}{(s-a)}\left(\frac{3a+7}{18a^2-98}\right) - \frac{1}{(s+a)}\left(\frac{3a-7}{18a^2-98}\right) - \frac{1}{(3s-7)}\left(\frac{42}{18a^2-98}\right). \tag{5.117}$$

Finally, by using the inverse Laplace transform tables, one inverts the above from $X(s)$ to $x(t)$. The answer is

$$x(t) = \left(\frac{3a+7}{18a^2-98}\right)\exp(at) - \left(\frac{3a-7}{18a^2-98}\right)\exp(-at) - \exp\left(\frac{7t}{3}\right)\left(\frac{14}{18a^2-98}\right). \tag{5.118}$$

Checking Accuracy of the Result $x(t)$

To that purpose, for $x(t)$ as per (5.118), one uses the homogeneous part of the parent differential equation (5.109), namely $3\,\frac{dx(t)}{dt} - 7x(t)$, and checks to see whether it equals the inhomogeneous part $cosh(at)$.

We get

$$\begin{aligned} &3\frac{dx(t)}{dt} - 7x(t) \\ &= 3a\left(\frac{3a+7}{18a^2-98}\right)\exp(at) + 3a\left(\frac{3a-7}{18a^2-98}\right)\exp(-at) \\ &- \left(\frac{98}{18a^2-98}\right)\exp\left(\frac{7}{3t}\right) - 7\left(\frac{3a+7}{18a^2-98}\right)\exp(at) \\ &+ 7\left(\frac{3a-7}{18a^2-98}\right)\exp(-at) + \left(\frac{98}{18a^2-98}\right)\exp\left(\frac{7}{3t}\right). \end{aligned} \tag{5.119}$$

Combining $\exp(at), \exp(-at)$, and the $\exp\left(\frac{7}{3t}\right)$ terms, (5.120) leads to the expression

$$\begin{aligned} &3\frac{dx(t)}{dt} - 7x(t) \\ &= \exp(at)\left[\frac{(3a+7)(3a-7)}{18a^2-98}\right] + \exp(-at)\left[\frac{(3a-7)(3a+7)}{18a^2-98}\right] \\ &- \left(\frac{98}{18a^2-98}\right)\exp\left(\frac{7}{3t}\right) + \left(\frac{98}{18a^2-98}\right)\exp\left(\frac{7}{3t}\right) \end{aligned} \tag{5.120}$$

with the result

$$3\,\frac{\mathrm{d}x(t)}{\mathrm{d}t} - 7\,x(t) = \left\{\frac{\exp(a\,t) + \exp(-a\,t)}{2}\right\} = cosh(a\,t). \qquad (5.121)$$

Q.E.D.

Solution: (3)

Another first-order differential equation with boundary condition $x(t) = 0$ for $t < 0$ is solved below.

$$(3) \;:\; 5\frac{\mathrm{d}x(t)}{\mathrm{d}t} + 4\,x(t) = sinh(a\,t)\;. \qquad (5.122)$$

Solution of (3).

Laplace transform of both sides of differential equation (5.122)—namely equation (3)—are

$$5\,\{s\,X(s) - x(0)\} + 4\,X(s) = \frac{a}{s^2 - a^2}\;. \qquad (5.123)$$

Or

$$X(s)\;(5\,s + 4) - 5\,x(0) = \frac{a}{s^2 - a^2}\;', \qquad (5.124)$$

Note: We have used the Laplace transform table provided in (5.78). To satisfy the initial boundary condition, set $x(0) = 0$. As a result, $X(s)$ becomes

$$X(s) = \left[\frac{a}{(5\,s + 4)\,(s - a)\,(s + a)}\right]\;. \qquad (5.125)$$

Notice how Laplace transform converts a function of some variable—say, t—into a function of another variable—say, s.

5.8.3 *Partial Fraction Decomposition of (5.125)*

To decompose (5.125) into partial fractions, express $X(s)$ as follows:

$$X(s) \equiv \frac{A}{(5\,s + 4)} + \frac{B}{(s - a)} + \frac{C}{(s + a)} \qquad (5.126)$$

Multiply the right-hand sides of (5.125) and (5.126)
by $(5\,s + 4)\,(s - a)\,(s + a)$ and equate the two results. [Note the objective here is to get a on the left-hand side of (5.127). We get

$$a = A\,(s - a)\,(s + a) + B\,(5\,s + 4)\,(s + a) + C\,(5\,s + 4)(s - a)\;. \qquad (5.127)$$

If (5.127) is to hold for arbitrary values of s, terms with any particular power of s must be equal on both sides of this equation. Accordingly, comparison of the s^2, s, and s^0 terms gives

$$\begin{aligned} A + 5B + 5C &= 0 \ ; \\ 5aB + 4B - 5aC + 4C &= 0 \ ; \\ -Aa^2 + 4aB - 4aC &= a \ . \end{aligned} \tag{5.128}$$

Equation (5.128) is readily solved and the result is

$$A = \left(\frac{5}{2}\right)\left[\frac{1}{4-5a} - \frac{1}{4+5a}\right] \ ; \ B = \frac{1}{2(4+5a)} \ ; \ C = -\frac{1}{2(4-5a)} \ . \tag{5.129}$$

By using (5.126) and (5.129), $X(s)$ is rewritten as

$$X(s) = \frac{20}{(5s+4)(16-25a^2)} + \frac{1}{(8+10a)(s-a)} - \frac{1}{(8-10a)(s+a)} \ . \tag{5.130}$$

Finally, by using the inverse Laplace transform tables, one inverts the above from $X(s)$ to $x(t)$. The result is

$$x(t) = \frac{4}{(16-25a^2)} \exp\left(-\frac{4t}{5}\right) + \frac{1}{(8+10a)} \exp(at) - \frac{1}{(8-10a)} \exp(-at) \ . \tag{5.131}$$

Checking Accuracy of the Result (5.131)

In order to check the accuracy of the result $x(t)$, given as per (5.131), one uses the homogeneous part of the parent differential equation (5.122), namely $5\,\frac{dx(t)}{dt} + 4\,x(t)$, and checks to see whether it equals the inhomogeneous part $sinh(a\,t)$.

To that end, first one calculates $5\,\frac{dx(t)}{dt}$.

$$\begin{aligned} 5\frac{dx(t)}{dt} = &-\left(\frac{16}{16-25a^2}\right)\exp\left(-\frac{4t}{5}\right) + \left(\frac{5a}{8+10a}\right)\exp(at) \\ &+ \left(\frac{5a}{8-10a}\right)\exp(-at) \ . \end{aligned} \tag{5.132}$$

Next one calculates $4\,x(t)$.

$$4\,x(t) = \left(\frac{16}{16 - 25\,a^2}\right) \exp\left(-\frac{4\,t}{5}\right) + \left(\frac{2}{4 + 5\,a}\right) \exp(a\,t) - \left(\frac{2}{4 - 5\,a}\right) \exp(-\,a\,t)\,. \tag{5.133}$$

Upon adding, (5.132) and (5.133) one can write

$$\begin{aligned} & 5\frac{\mathrm{d}x(t)}{\mathrm{d}t} + 4\,x(t) \\ &= \left(\frac{5\,a}{8 + 10\,a}\right) \exp(a\,t) + \left(\frac{4}{8 + 10\,a}\right) \exp(a\,t) \\ &+ \left(\frac{5\,a}{8 - 10\,a}\right) \exp(-\,a\,t) - \left(\frac{4}{8 - 10\,a}\right) \exp(-\,a\,t) \end{aligned} \tag{5.134}$$

Equation (5.134) leads readily to the desired result.

$$\begin{aligned} & 5\frac{\mathrm{d}x(t)}{\mathrm{d}t} + 4\,x(t) \\ &= \frac{1}{2}\left\{\exp(a\,t) - \exp(-\,a\,t)\right\} \;=\; sinh(a\,t). \end{aligned} \tag{5.135}$$

Q.E.D.

Solution: (4)

Another first-order differential equation with boundary condition $x(t) = 0$ for $t < 0$ is solved below.

$$(4)\;:\; 4\frac{\mathrm{d}x(t)}{\mathrm{d}t} + 9\,x(t) \;=\; \exp(a\,t)\,sinh(a\,t)\;. \tag{5.136}$$

Solution of (4).

Laplace transform of both sides of differential equation (5.136)—namely equation (4)—are

$$4\,\{s\,X(s) - x(0)\} + 9\,X(s) \;=\; \frac{a}{s\,(s - 2\,a)}\,. \tag{5.137}$$

Or

$$X(s)\;(4\,s + 9) - 4\,x(0) \;=\; \frac{a}{s\,(s - 2\,a)}\,' \tag{5.138}$$

Note: We have used the Laplace transform table provided in (5.78). To satisfy the initial boundary condition, set $x(0) = 0$. As a result, $X(s)$ becomes

$$X(s) = \left[\frac{a}{(s)\,(s-2\,a)\,(4\,s+9)} \right] . \tag{5.139}$$

5.8.4 Partial Fraction Decomposition of (5.139)

To decompose (5.139) into partial fractions, express $X(s)$ as follows:

$$X(s) \equiv \frac{A}{(s)} + \frac{B}{(s-2\,a)} + \frac{C}{(4\,s+9)} \tag{5.140}$$

Multiply the right-hand sides of (5.139) and (5.140)
by $(s)\,(s-2\,a)\,(4\,s+9)$ and equate the two results. [Note the objective here is to get a on the left-hand side of (5.141). We get

$$\begin{aligned} a &= A\,(s-2\,a)\,(4\,s+9) + B\,(s)\,(4\,s+9) + C\,(s)\,(s-2\,a) \\ &= A\left[4\,s^2 + 9\,s - 8\,a\,s - 18\,a\right] + B\left[4\,s^2 + 9\,s\right] + C\left[s^2 - 2\,a\,s\right] . \end{aligned} \tag{5.141}$$

If (5.141) is to hold for arbitrary values of s, terms with any particular power of s must be equal on both sides of this equation. Accordingly, comparison of the s^2, s, and s^0 terms gives

$$\begin{aligned} 4\,A + 4\,B + C &= 0 \ ; \\ 9\,A - 8\,a\,A + 9\,B - 2\,a\,C &= 0 \ ; \\ -18\,a\,A &= a \ . \end{aligned} \tag{5.142}$$

Equation (5.142) is readily solved and the result is

$$A = -\left(\frac{1}{18}\right) \ ; \ B = \left(\frac{1}{18+16\,a}\right) \ ; \ C = -\left(\frac{2}{9+8\,a}\right) + \left(\frac{4}{18}\right) . \tag{5.143}$$

By using (5.140) and (5.143), $X(s)$ is rewritten as

$$X(s) = \left(\frac{-1}{18\,s}\right) + \left(\frac{1}{18+16\,a}\right)\left(\frac{1}{s-2\,a}\right) + \left[\frac{4}{18} - \frac{2}{9+8\,a}\right]\left(\frac{1}{4\,s+9}\right) . \tag{5.144}$$

Finally, by using the inverse Laplace transform tables, one inverts $X(s)$ to $x(t)$. The result is

$$\begin{aligned} x(t) = &-\left(\frac{1}{18}\right) + \left[\frac{\exp(2\,a\,t)}{18+16\,a}\right] \\ &+ \left[\frac{4}{18} - \frac{2}{9+8\,a}\right]\left(\frac{1}{4}\right)\exp\left(-\frac{9\,t}{4}\right) . \end{aligned} \tag{5.145}$$

Checking Accuracy of the Result (5.145)

In order to check the accuracy of the result $x(t)$, given as per (5.145), one uses the homogeneous part of the parent differential equation (5.136), namely $4\,\frac{\mathrm{d}x(t)}{\mathrm{d}t} + 9\,x(t)$, and checks to see whether it equals the inhomogeneous part $\exp(a\,t)$ $sinh(a\,t)$. To that end, first one calculates $4\,\frac{\mathrm{d}x(t)}{\mathrm{d}t}$.

$$4\,\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \left[\frac{4\,a\,\exp(2\,a\,t)}{9+8\,a}\right] - \left(\frac{9}{4}\right)\left[\frac{2}{9} - \frac{2}{9+8\,a}\right]\exp\left(-\frac{9\,t}{4}\right). \quad (5.146)$$

And next $9\,x(t)$.

$$9\,x(t) = -\left(\frac{1}{2}\right) + \left(\frac{9}{2}\right)\left[\frac{\exp(2\,a\,t)}{(9+8\,a)}\right] + \left(\frac{9}{4}\right)\left[\frac{2}{9} - \frac{2}{9+8\,a}\right]\exp\left(-\frac{9\,t}{4}\right). \quad (5.147)$$

Upon adding (5.146) and (5.147), one gets

$$4\frac{\mathrm{d}x(t)}{\mathrm{d}t} + 9\,x(t) = \left[\frac{4\,a\,\exp(2\,a\,t)}{9+8\,a}\right] - \left(\frac{1}{2}\right) + \left(\frac{9}{2}\right)\left[\frac{\exp(2\,a\,t)}{(9+8\,a)}\right]. \quad (5.148)$$

Equation (5.148) leads readily to the desired result.

$$4\frac{\mathrm{d}x(t)}{\mathrm{d}t} + 9\,x(t) \;=\; \frac{\exp\,(2\,a\,t) - 1}{2}\;. \quad (5.149)$$

Q.E.D.

Solution: (5)

Another first-order differential equation with boundary condition $x(t) = -6$ for $t < 0$ is solved below.

$$(5)\;:\;\frac{\mathrm{d}x(t)}{\mathrm{d}t} + 5\,x(t) \;=\; \exp(7\,t)\,. \quad (5.150)$$

Solution of (5).

Laplace transform of both sides of differential equation (5.150)—namely equation (5)—are

$$\{s\,X\,(s) - x(0)\} + 5\,X\,(s) \;=\; \frac{1}{(s-7)}\,. \quad (5.151)$$

Note: We have used the Laplace transform table provided in (5.78) and to satisfy the initial boundary condition, we have set $x(0) = -6$. As a result, $X\,(s)$ becomes

$$X\,(s)\;(s+5) + 6 \;=\; \frac{1}{(s-7)}\,. \quad (5.152)$$

Equivalently,

$$X(s) \;=\; \left[\frac{1}{(s+5)\,(s-7)}\right] - \left(\frac{6}{s+5}\right) \;=\; \left[\frac{43-6\,s}{(s+5)\,(s-7)}\right] . \quad (5.153)$$

5.8.5 *Partial Fraction Decomposition of (5.153)*

To decompose (5.153) into partial fractions, express $X(s)$ as follows:

$$X(s) \equiv \frac{A}{(s+5)} + \frac{B}{(s-7)} . \quad (5.154)$$

Multiply the right-hand sides of (5.153) and (5.154) by $(s+5)\,(s-7)$ and equate the two results. We get

$$43 - 6\,s \;=\; A\,(s-7) + B\,(s+5) \;=\; s\,(A+B) + (5\,B - 7\,A) . \quad (5.155)$$

If (5.155) is to hold for arbitrary values of s, terms with any particular power of s must be equal on both sides of this equation. Accordingly, comparison of the s and s^0 terms gives

$$\begin{aligned} A + B \;&=\; -\,6 \;; \\ -7\,A + 5\,B \;&=\; 43 \end{aligned} \quad (5.156)$$

leading to

$$A \;=\; -\left(\frac{73}{12}\right) \;;\; B \;=\; \left(\frac{1}{12}\right) . \quad (5.157)$$

With the help of (5.154) and (5.157), $X(s)$ is rewritten as

$$X(s) = \left(\frac{-73}{12\,(s+5)}\right) + \left(\frac{1}{12}\right)\left(\frac{1}{s-7}\right) . \quad (5.158)$$

Finally, by using the inverse Laplace transform tables, one inverts $X(s)$ to $x(t)$. The result is

$$x(t) = -\left(\frac{73}{12}\right)\exp(-5\,t) + \left(\frac{1}{12}\right)\exp(7\,t) . \quad (5.159)$$

Checking Accuracy of the Result (5.159)

The result $x(t)$, as given in (5.159), is accurate because, it satisfies the primary differential equation (5.150). To check proceed as follows:

$$\begin{aligned} &\frac{\mathrm{d}x(t)}{\mathrm{d}t} + 5\,x(t) \\ &= 5\left(\frac{73}{12}\right)\exp(-5\,t) + \left(\frac{7}{12}\right)\exp(7\,t) \\ &- 5\left(\frac{73}{12}\right)\exp(-5\,t) + \left(\frac{5}{12}\right)\exp(7\,t) \;=\; \exp(7\,t)\,. \end{aligned} \tag{5.160}$$

Q.E.D.

5.9 Second-Order Differential Equations

5.9.1 Solution by Laplace transform

By the use of Laplace transform, solve the following second-order differential equation

$$(I)\;:\quad 2\,y''(t) + 3\,y'(t) - 2\,y(t) \;=\; t\,\exp(-2\,t) \tag{5.161}$$

with the boundary condition

$$y(0) \;=\; 0 \quad \textit{and} \quad y'(0) \;=\; -2 \quad . \tag{5.162}$$

Solution: (I)

Laplace transform of both sides of differential equation (5.161)—namely equation (I)—are

$$2\left\{s^2\,Y(s) - s\,y(0) - y'(0)\right\} + 3\left\{s\,Y(s) - y(0)\right\} - 2\,Y(s) \;=\; \frac{1}{(s+2)^2}\,. \tag{5.163}$$

Upon inserting the boundary condition (5.162), (5.163) becomes

$$Y(s)\left[2\,s^2 + 3\,s - 2\right] \;=\; Y(s)\left[(2\,s-1)\,(s+2)\right] \;=\; \frac{1}{(s+2)^2} - 4. \tag{5.164}$$

Equation (5.164) can be rewritten in a more compact form as

$$Y(s) \;=\; \frac{-4\,s^2 - 16\,s - 15}{(2\,s-1)\,(s+2)^3}\,. \tag{5.165}$$

5.9.2 *Partial Fraction Decomposition of (5.165)*

To decompose (5.165) into partial fractions, express $Y(s)$ as follows:

$$Y(s) \equiv \frac{A}{(2\,s-1)} + \frac{B}{(s+2)} + \frac{C}{(s+2)^2} + \frac{D}{(s+2)^3} \tag{5.166}$$

Multiply the right-hand sides of (5.165) and (5.166)
by $(2\,s-1)\,(s+2)^3$ and equate the two results. We get

$$\begin{aligned} &-4\,s^2 - 16\,s - 15 \\ &= A\,(s+2)^3 + B\,(2\,s-1)\,(s+2)^2 + C\,(2\,s-1)\,(s+2) + D\,(2\,s-1) \\ &= (A+2\,B)\,s^3 + (6\,A+7\,B+2\,C)\,s^2 + (12\,A+4\,B+3\,C+2\,D)\,s \\ &+ (8\,A-4\,B-2\,C-D)\,. \end{aligned} \tag{5.167}$$

For (5.167) to hold for arbitrary values of s, terms with any particular power of s must be equal on both sides of this equation. Accordingly, comparison of the s^3, s^2, s, and s^0 terms gives

$$\begin{aligned} A + 2\,B &= 0\,; \\ 6\,A + 7\,B + 2\,C &= -4\,; \\ 12\,A + 3\,C + 4\,B - 2\,D &= -16\,; \\ 8\,A - 4\,B - 2\,C - D &= -15\,. \end{aligned} \tag{5.168}$$

Equation (5.168) is straightforward to solve and the result is

$$A = -\frac{192}{125}\;;\; B = \frac{96}{125}\;;\; C = -\frac{10}{125}\;;\; D = -\frac{25}{125}\,. \tag{5.169}$$

There is a common denominator of 125 for A, B, C, and D in (5.169). Therefore upon plugging these results for A, B, C, and D into the Laplace transform (5.166) one gets

$$Y(s) = \frac{1}{125}\left[-\frac{96}{(s-\frac{1}{2})} + \frac{96}{(s+2)} - \frac{10}{(s+2)^2} - \frac{25}{(s+2)^3}\right]. \tag{5.170}$$

Finally, by using the inverse Laplace transform tables, one inverts the above from $Y(s)$ to $y(t)$. The result is

$$x(t) = \frac{1}{125}\left[-96\exp\left(\frac{t}{2}\right) + 96\,\exp(-2\,t) - 10\,t\,\exp(-2\,t) - \frac{25}{2}\,t^2\exp(-2\,t)\right] \tag{5.171}$$

Checking Accuracy of the Result (5.171)

In order to confirm the accuracy of $x(t)$, as per (5.171), one uses first the homogeneous part of the parent differential equation (5.161), namely $2\,y''(t) + 3\,y'(t) - 2\,y(t)$, and checks to see whether after using the boundary condition $y(0) = 0$ and $y'(0) = -2$ it equals the inhomogeneous part of the parent differential equation, i.e., $t\,\exp(-2\,t)$. To that purpose, let us start in seriatim with the first two terms of $x(t)$ recorded in (5.171). That is,

$$\begin{aligned} &\frac{96}{125}\left[2\,\frac{\mathrm{d}^2}{\mathrm{d}t^2} + 3\frac{\mathrm{d}}{\mathrm{d}t} - 2\right].\{-\exp(t/2) + \exp(-2\,t)\} \\ &= \frac{96}{125}\left\{\left(-\frac{2}{4} - \frac{3}{2} + 2\right)\exp(t/2) + (4 - 6 - 2)\exp(-2\,t)\right\} \;=\; 0\;. \end{aligned} \tag{5.172}$$

Next, we examine the response of the third and the fourth terms in (5.171). Meaning the expressions in the following two equations:

$$-\frac{10}{125}\left[2\,\frac{\mathrm{d}^2}{\mathrm{d}t^2} + 3\frac{\mathrm{d}}{\mathrm{d}t} - 2\right][t\exp(-2\,t)]\,, \tag{5.173}$$

$$-\frac{25}{2\times 125}\left[2\,\frac{\mathrm{d}^2}{\mathrm{d}t^2} + 3\frac{\mathrm{d}}{\mathrm{d}t} - 2\right][t^2\,\exp(-2\,t)]\,. \tag{5.174}$$

With a little bit of simple differentiation and great deal of straightforward algebra, (5.173) and (5.174) yield the results $\frac{2}{5}\exp(-2\,t)$ and $\exp(-2\,t)\left[t - \frac{2}{5}\right]$. Adding the two gives $t\,\exp(-2\,t)$. Q.E.D.

(II)

With the help of Laplace transform, solve the following second-order differential equation

$$(II)\;:\quad y''(t) - 6\,y'(t) + 9\,y(t) \;=\; 2\,sin(3\,t). \tag{5.175}$$

The boundary condition is:

$$y(0) \;=\; -1 \quad and \quad y'(0) \;=\; -4\quad. \tag{5.176}$$

Solution: (II)

Laplace transform of both sides of differential equation (5.175)—namely equation (II)—are

$$\left\{s^2\,Y(s) - s\,y(0) - y'(0)\right\} - 6\,\{s\,Y(s) - y(0)\} + 9\,Y(s) \;=\; \frac{6}{(s^2+9)}\,. \tag{5.177}$$

Upon inserting the boundary condition (5.176), (5.177) becomes

$$Y(s)\left[s^2+6s+9\right]+s-2 \;=\; \frac{6}{s^2+9} \tag{5.178}$$

After a little bit of algebra equation (5.178) can be rewritten in a more compact form.

$$Y(s) \;=\; \frac{-s^3+2s^2-9s+24}{(s^2+9)\,(s-3)^2}\,. \tag{5.179}$$

5.9.3 *Partial Fraction Decomposition of (5.179)*

To decompose (5.179) into partial fractions, express $Y(s)$ as follows:

$$Y(s) \equiv \frac{As+B}{(s^2+9)} + \frac{C}{(s-3)} + \frac{D}{(s-3)^2}\,. \tag{5.180}$$

Multiply the right-hand sides of (5.179) and (5.180)
by $(s^2+9)\,(s-3)^2$ and equate the two results. We get

$$-s^3+2s^2-9s+24 \;=\; (As+B)(s-3)^2 + C\,(s^2+9)\,(s-3) + D\,(s^2+9)\,. \tag{5.181}$$

For (5.181) to hold for arbitrary values of s, terms with any particular power of s must be equal on both sides of this equation. Accordingly, comparison of the s^3, s^2, s, and s^0 terms gives

$$\begin{aligned} A+C &= -1\;; \\ -6A+B-3C+D &= 2\;; \\ -6B+9A+9C &= -9\;; \\ 9B-27C+9D &= 24\,. \end{aligned} \tag{5.182}$$

Equation (5.182) is straightforward to solve and the result is

$$A \;=\; \frac{1}{9}\;;\;\; B \;=\; 0\;;\;\; C \;=\; -\frac{10}{9}\;;\;\; D \;=\; -\frac{6}{9}\,. \tag{5.183}$$

While $B=0$, there is a common denominator of 9 for A, C, and D in (5.183). Therefore upon plugging these results for A, B, C, and D into the Laplace transform (5.180) one gets

$$Y(s) \;=\; \frac{1}{9}\left[\frac{s}{(s^2+9)} - \frac{10}{(s-3)} - \frac{6}{(s-3)^2}\right]\,. \tag{5.184}$$

Finally, by using the inverse Laplace transform tables, one inverts the above from $Y(s)$ into $Y(t)$. The result is

$$Y(t) = \frac{1}{9}\left[cos(3\,t) - 10\,\exp(3\,t) - 6\,t\,\exp(3\,t)\right]\,. \tag{5.185}$$

Checking Accuracy of the Result (5.185)

In order to confirm the accuracy of $Y(t)$, as per (5.185), we use the parent differential equation (5.175), namely $Y''(t) - 6\,Y'(t) + 9\,Y(t)$, and check to see whether after using the boundary condition $y(0) = -1$ and $y'(0) = -4$ it equals the inhomogeneous part of the parent differential equation, namely $2\,sin(3\,t)$. It does and the relevant algebra is described below.

$$\begin{aligned}
&Y''(t) - 6\,Y'(t) + 9\,Y(t)\\
&= -\,cos(3\,t) - 14\,\exp(3\,t) - 6\,t\,\exp(3\,t)\\
&-6\left[-\frac{3}{9}\,sin(3\,t) - \frac{10}{3}\,\exp(3\,t) - \frac{2}{3}\,\exp(3\,t) - 2\,t\,\exp(3\,t)\right]\\
&+\,cos(3\,t) - 10\,\exp(3\,t) - 6\,t\,\exp(3\,t)\,.\\
&= 2\,sin(3\,t)\,.
\end{aligned} \tag{5.186}$$

Q.E.D.

With the help of Laplace transform, solve the following IVP.

$$(III)\;:\quad y''(t) - 8\,y'(t) + 7\,y(t) \;=\; 9\,t. \tag{5.187}$$

The boundary condition is

$$y(0) \;=\; -1 \quad,\quad y'(0) \;=\; 2 \quad. \tag{5.188}$$

Solution: (III)

Laplace transform of both sides of differential equation (5.187)—namely equation (III)—are

$$\left\{s^2\,Y(s) - s\,y(0) - y'(0)\right\} - 8\,\{s\,Y(s) - y(0)\} + 7\,Y(s) \;=\; \frac{9}{s^2}\,. \tag{5.189}$$

Upon inserting the boundary condition (5.188), (5.189) becomes

$$Y(s)\,\left[s^2 - 8\,s + 7\right] + s - 10 \;=\; \frac{9}{s^2} \tag{5.190}$$

Equation (5.190) can readily be rewritten in a more compact form.

$$Y(s) \;=\; \frac{-s^3 + 10\,s^2 + 9}{s^2\,(s-7)\,(s-1)}\,. \tag{5.191}$$

5.9.4 *Partial Fraction Decomposition of (5.191)*

To decompose (5.191) into partial fractions, we express $Y(s)$ as follows:

$$Y(s) \equiv \frac{A}{s} + \frac{B}{s^2} + \frac{C}{(s-7)} + \frac{D}{(s-1)}\,. \tag{5.192}$$

Multiply the right-hand sides of (5.191) and (5.192)
by $s^2\,(s-7)\,(s-1)$ and equate the two results. We get

$$\begin{aligned} -s^3 + 10\,s^2 + 9 = {} & A\,s\,(s-7)\,(s-1) \;+\; B\,(s-7)\,(s-1) \\ & +\, C\,s^2\,(s-1) \;+\; D\,s^2\,(s-7)\,. \end{aligned} \tag{5.193}$$

For (5.193) to hold for arbitrary values of s, terms with any particular power of s must be equal on both sides of this equation. Accordingly, comparison of the s^3, s^2, s, and s^0 terms gives

$$\begin{aligned} A + C + D &= -1\;; \\ -8A + B - C &= 10\;; \\ 7A - 8B - 7D &= 0\;; \\ 7B &= 9\,. \end{aligned} \tag{5.194}$$

Equation (5.194) is straightforward to solve and the result is

$$A \;=\; \frac{72}{49}\;;\;\; B \;=\; \frac{9}{7}\;;\;\; C \;=\; \frac{26}{49}\;;\;\; D \;=\; -3\,. \tag{5.195}$$

Upon plugging the results for A, B, C, and D into the Laplace transform (5.192) one gets

$$Y(s) = \frac{\frac{72}{49}}{s} + \frac{\frac{9}{7}}{s^2} + \frac{\frac{26}{49}}{(s-7)} - \frac{3}{(s-1)}\,. \tag{5.196}$$

Finally, by using the inverse Laplace transform tables, one inverts the above from $Y(s)$ to $Y(t)$. The result is

$$Y(t) = \frac{72}{49} + \left(\frac{9}{7}\right) t + \left(\frac{26}{49}\right) \exp(7\,t) - 3\exp(t). \tag{5.197}$$

Checking Accuracy of the Result (5.197)

In order to confirm the accuracy of $Y(t)$, as per (5.197), we use the parent differential equation (5.187), namely $Y''(t) - 8\,Y'(t) + 7\,Y(t)$, and check to see whether after using the boundary condition $y(0) = -1$ and $y'(0) = 2$, it equals the inhomogeneous part of the parent differential equation, namely $9\,t$. It does, and the relevant algebra is displayed neatly below.

$$\begin{aligned} & Y''(t) - 8\,Y'(t) + 7\,Y(t) \\ &= \left[26\,\exp(7\,t) - 3\,\exp(t)\right] \\ &+ \left[-\left(\frac{72}{7}\right) - \left(\frac{8\,\times\,26}{7}\right)\exp(7\,t) + 24\,\exp(t)\right] \\ &+ \left[\left(\frac{72}{7}\right) + 9\,t + \left(\frac{26}{7}\right)\exp(7\,t) - 21\,\exp(t)\right] . \\ &= 9\,t\,. \end{aligned} \tag{5.198}$$

Q.E.D.

With the help of Laplace transform, solve the following differential equation.

$$(IV)\ :\quad 2\,y''(t) - 5\,y'(t) - 3\,y(t) \;=\; t\,\exp(-t). \tag{5.199}$$

The boundary condition is

$$y(0) \;=\; 1 \quad , \quad y'(0) \;=\; 2 \quad . \tag{5.200}$$

Solution: (IV)

Laplace transform of both sides of differential equation (5.199)—namely equation (IV)—are

$$2\,\left\{s^2\,Y(s) - s\,y(0) - y'(0)\right\} - 5\,\{s\,Y(s) - y(0)\} - 3\,Y(s) \;=\; \frac{1}{(s+1)^2}\,. \tag{5.201}$$

Upon inserting the boundary condition (5.200), (5.201) becomes

$$Y(s)\,\left[2\,s^2 - 5\,s - 3\right] - 2\,s - 4 + 5 \;=\; \frac{1}{(s+1)^2} \tag{5.202}$$

Equation (5.202) can readily be rewritten in a more compact form.

$$Y(s) \;=\; \frac{(2\,s-1)\,(s+1)^2\,+1}{(s+1)^2\,(2\,s+1)\,(s-3)}\,. \tag{5.203}$$

5.9.5 *Partial Fraction Decomposition of (5.203)*

To decompose (5.203) into partial fractions, express $Y(s)$ as follows:

$$Y(s) \equiv \frac{A}{(2\,s+1)} + \frac{B}{(s-3)} + \frac{C}{(s+1)} + \frac{D}{(s+1)^2}\,. \tag{5.204}$$

Multiply the right-hand sides of (5.203) and (5.204)
by $(s+1)^2\,(2\,s+1)\,(s-3)$ and equate the two results. As usual do the relevant algebra and determine the parameters A, B, C, etc. [Note : For instance, compare with (5.192) to (5.195).] We get

$$A = \left(\frac{236}{7\,\times\,16}\right)\;;\; B = \left(\frac{1}{7\,\times\,16}\right)\;;\; C = \left(\frac{9}{16}\right)\;;\; D = \left(\frac{1}{4}\right)\,. \tag{5.205}$$

Upon plugging the above results for $A, B, C,$ and D into the Laplace transform (5.204) one gets

$$\begin{aligned} Y(s) = &\left(\frac{59}{28}\right)\frac{1}{(2\,s+1)} + \left(\frac{1}{7\,\times\,16}\right)\frac{1}{(s-3)} \\ &+ \left(\frac{9}{16}\right)\frac{1}{(s+1)} + \left(\frac{1}{4}\right)\frac{1}{(s+1)^2}\,. \end{aligned} \tag{5.206}$$

Finally, by using the inverse Laplace transform tables, one inverts the above from $Y(s)$ to $Y(t)$. The result is

$$\begin{aligned} Y(t) = &\left(\frac{59}{28}\right)\left(\frac{1}{2}\right)\exp(-t/2) + \left(\frac{1}{7\,\times\,16}\right)\exp(3\,t) \\ &+ \left(\frac{9}{16}\right)\exp(-t) + \left(\frac{t}{4}\right)\exp(-t)\;. \end{aligned} \tag{5.207}$$

Checking Accuracy of the Result (5.207)

In order to determine the accuracy of $Y(t)$, as per (5.207), we use the parent differential equation (5.201), namely $2\,Y''(t) - 5\,Y'(t) - 3\,Y(t)$, and check to see whether after using the boundary condition $y(0) = 1$ and $y'(0) = 2$, the result equals the inhomogeneous part of the parent differential equation, namely $t\,\exp(-t)$. We find that it does, and the relevant algebra is displayed in seriatim below in (5.208), (5.209), and (5.210).

$$\begin{aligned} 2\,Y''(t) = &\left(\frac{1}{4}\right)\left(\frac{59}{28}\right)\exp(-t/2) + \left(\frac{9}{56}\right)\exp(3\,t) + \left(\frac{9}{8}\right)\exp(-t) \\ &-\;\exp(-t) + \left(\frac{1}{2}\right)t\,\exp(-t)\,. \end{aligned} \tag{5.208}$$

$$-5\,Y'(t) = \left(\frac{5}{4}\right)\left(\frac{59}{28}\right)\exp(-t/2) - \left(\frac{15}{7\times 16}\right)\exp(3\,t) + \left(\frac{45}{16}\right)\exp(-t) - \left(\frac{5}{4}\right)\exp(-t) + \left(\frac{5}{4}\right)\,t\,\exp(-t)\,. \tag{5.209}$$

$$-3\,Y(t) = -3\left(\frac{59}{56}\right)\exp(-t/2) - \left(\frac{3}{7\times 16}\right)\exp(3\,t) - \left(\frac{27}{16}\right)\exp(-t) - \left(\frac{3}{4}\right)\,t\,\exp(-t)\,. \tag{5.210}$$

Q.E.D.

Adding (5.208), (5.209), and (5.210) leads to the desired result $t\,\exp(-t)$. To double check this statement, examine the details of the addition as given below. We have

$$\begin{aligned}
&\exp(-t/2)\left(\frac{59}{56}\right)\left[\frac{1}{2}+\frac{5}{2}-3\right] \;=\; 0\,.\\
&\exp(3\,t)\left[\left(\frac{9}{56}\right)-\left(\frac{15}{7\times 16}\right)-\left(\frac{3}{7\times 16}\right)\right] \;=\; 0\,.\\
&\exp(-t)\left(\frac{1}{16}\right)[18-16+45-20-27] \;=\; 0\,.\\
&t\,\exp(-t)\left(\frac{1}{4}\right)[2+5-3] \;=\; t\,\exp(-t)\,.
\end{aligned} \tag{5.211}$$

'Quod Erat Demonstrandum.'

5.10 Need for Convolution

We have studied inhomogeneous linear ordinary differential equations of the form

$$O(t)\,P(t) = \Gamma(t) \tag{5.212}$$

where the differential operator $O(t)$, the solution $P(t)$, as well as the inhomogeneous term represented by a function $\Lambda(t)$, all involved constants and derivatives with respect to a single variable t. In addition, there were some specified boundary conditions that the solution had to satisfy. The important thing to note was that the inhomogeneous term $L(t)$ was not arbitrary. Rather, it was properly defined.

In preceding chapters, various methods for solving homogeneous linear ordinary differential equations were discussed.

For inhomogeneous linear ordinary differential equations, those methods required working out the particular integral, $I_{pi}(t)$, ab initio for every different inhomogeneous term $\Gamma(t)$. In this regard, Green's function methodology appeared as a savior, for the reason that Green's function—if successfully calculated—once and for all, helps solve the differential equation for arbitrary values of the inhomogeneous term $\Gamma(t)$, . Consequently, for any particular differential operator $O(t)$, in addition to the homogeneous solution, only Green's function itself needs to be worked out.

Unfortunately, an appropriate Green's function sometimes is hard to work out. And even when it is not, the solution is likely to contain infinite terms and be very inconvenient to use.

For instance, the eigenfunction Green's function leads to infinite series that are often both difficult to work out and complicated to work with. Even the closed-form solutions of Green's function, that make use of Dirac's delta function, are complicated to calculate and can be employed only to a limited class of Green's functions.

With the hope of remedying these difficulties, one uses convolution integrals which are useful for solving differential equations with arbitrary values of the inhomogeneous term—namely the forcing function—$\Gamma(t)$.

5.11 Convolution Integral

Given continuous, or at least piecewise continuous, functions $f(t)$ and $g(t)$ on $[0, \infty]$, the convolution integral of $f(t)$ and $g(t)$ is defined as

$$f(t)\,(*)g(t) = \int_0^t f(t-\tau)\,g(\tau)\,\mathrm{d}\tau \;=\; \int_0^t g(t-\tau)f(\tau)\,\mathrm{d}\tau \;. \tag{5.213}$$

It is helpful to observe how a convolution integral is employed for solving differential equations with an arbitrary inhomogeneous term . To that end, let us solve the following very simple IVP.

$$y''(t) + y(t) \;=\; \gamma(t) \;. \tag{5.214}$$

Its Laplace transform is

$$Y(s)\left[s^2 - s\,y(0) - y'(0)\right] + Y(s) \;=\; \gamma(s) \tag{5.215}$$

Using the boundary condition

$$y(t=0) \;=\; 1 \;,\;\; y'(t=0) \;=\; -\,6 \;, \tag{5.216}$$

(5.215) leads to

$$Y(s) \;=\; \frac{s}{(s^2+1)} - \frac{6}{(s^2+1)} + \frac{\gamma(s)}{(s^2+1)} \;. \tag{5.217}$$

Finally, taking the inverse transform of (5.217) one gets

$$y(t) \;=\; cos(t) \;-\; 6\,sin(t) + \int_0^t sin(\tau)\,\gamma(t-\tau)\,\mathrm{d}\tau \;. \tag{5.218}$$

The main idea of this exercise is that the above result applies to arbitrary choices of $\gamma(t)$. Just to make sure that we have not made an error somewhere let us make an extremely simple choice for the arbitrary $\gamma(t)$. That is, $\gamma(t) \;=\; t$. Then, (5.218) gives

$$y(t) = cos(t) \;-\; 6\,sin(t) + \left[\int_0^t sin(\tau).\,(t-\tau)\,\mathrm{d}\tau\right] \;. \tag{5.219}$$

Now integrate the following by parts

$$\int_0^t sin(\tau).\,(t-\tau)\,\mathrm{d}\tau \;. \tag{5.220}$$

One gets

$$\begin{aligned}\int_0^t sin(\tau).\,(t-\tau)\,\mathrm{d}\tau &= [-cos(\tau)\,(t\;-\;\tau)]_0^t - \left\{\int_0^t cos(\tau)\,\mathrm{d}\,\tau\right\} \\ &\;-\; cos(t)\,[t-t] \;+\; cos(0)\,[t-0] \;-\; [sin(t) - sin(0)] \\ &= t \;-\; sin(t) \;. \end{aligned} \tag{5.221}$$

Equations (5.219) to (5.222) lead to the final result.

$$\begin{aligned} y(t) &= cos(t) \;-\; 6\,sin(t) + [t \;-\; sin(t)] \;. \\ &= cos(t) \;-\; 7\,sin(t) \;+\; t. \end{aligned} \tag{5.222}$$

One can quickly confirm that this result is correct because

$$\begin{aligned} y''(t) + y(t) \;&=\; -\,cos(t) \;+\; 7\,sin(t) \;+\; cos(t) - 7\,sin(t) + t \\ = \gamma(t) \;&=\; t. \end{aligned} \tag{5.223}$$

Chapter 6
Special Types of Differential Equations



One special type of differential equation, namely the *Bernoulli*[1.] equation, was discussed in Chap. 4. [Compare, for instance, (4.19)–(4.77).] Here, that analysis is extended to other special type equations.

Included in this presentation are the *Clairaut*[2.] equations—[Compare (6.2)–(6.13)]—*Lagrange*[3.] equation—[Compare (6.19)–(6.31)], the separable equations—[Compare (6.32)–(6.35)], and the $\frac{dy}{dx} = \Phi\left(\frac{y}{x}\right)$ equations—[Compare (6.36)–(6.73)]. In addition, there are the so-called exact [Compare (6.74)–(6.91)] and inexact equations—[Compare (6.92)–(6.241)]—*Riccati*[4.] equations—[Compare (6.242)–(6.268)]—*Euler*[7.] equations—[Compare (6.269)–(6.315)], and the factorable equations—[Compare (6.316)–(6.344)].

Notation

Occasionally, for convenience, the following notation will be used.

$$\begin{aligned} q &= \frac{dy}{dx} \quad ; \\ q\,dx &= dy \quad . \end{aligned} \tag{6.1}$$

6.1 Clairaut Equation: Description

Consider an equation

$$y = \sigma x + f(\sigma) \tag{6.2}$$

where y is the dependent, x is the independent variable, and $f(\sigma)$ is a function of an arbitrary parameter σ. Differentiate (6.2) with respect to x,

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$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sigma \,,$$

and use (6.1) in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = q \,,$$

that leads to

$$\sigma = q \ .$$

Now substitute the variable q for the parameter σ in (6.2). This process leads to Clairaut equation

$$y = q\,x + f(q) \ . \tag{6.3}$$

Its general solution has already been recorded as (6.2). It is a straight line in the (x, y) plane that is obtained by replacing the slope, q, by its observed value σ. Indeed, the general solution is a single arbitrary parameter representation of a whole family of straight lines.

6.1.1 Solving Clairaut Equation

Consider the Clairaut equation (6.3). Differentiate both sides.

$$\mathrm{d}y = q\,\mathrm{d}x + x\,\mathrm{d}q + \mathrm{d}\,f(q) \ .$$

According to (6.1), $q\,\mathrm{d}x = \mathrm{d}y$. Thus, $\mathrm{d}y$ and $q\,\mathrm{d}x$ can be eliminated from the above differential equation. The result is

$$x\,\mathrm{d}q + \mathrm{d}\,f(q) = 0 \ . \tag{6.4}$$

It is convenient to introduce the notation

$$\mathrm{d}\,f(q) = F(q)\,\mathrm{d}q \ , \tag{6.5}$$

whereby (6.4) can be written as

$$[x + F(q)] \cdot \mathrm{d}q = 0 \ . \tag{6.6}$$

Notice that (6.6) is a product of two differential equations:

$$\begin{aligned} (1) \; &: \; \mathrm{d}q \; = \; 0 \; ; \\ (2) \; &: \; [x \; + \; F(q)] \; = \; 0 \; . \end{aligned} \tag{6.7}$$

The differential equation (6.7)-(1) is straightforward and can immediately be integrated. Initial integration gives

$$\int \mathrm{d}q \; = \; q \; = \; \int 0 \; = \; \sigma \; ,$$

where σ is an arbitrary constant. The next integration yields

$$\begin{aligned} \int q \, \mathrm{d}x \; &= \; \int \frac{\mathrm{d}y}{\mathrm{d}x} \, \mathrm{d}x \; = \; \int \mathrm{d}y \; = \; y \\ = \; \int \sigma \, \mathrm{d}x \; &= \; \sigma \, x + constant \; , \end{aligned}$$

which can be expressed as

$$y = \sigma \, x + f \, (\sigma) \; , \tag{6.8}$$

where $f \, (\sigma)$ is an arbitrary function of the arbitrary constant σ. Written in this form, y is the general solution and is identical to the solution expressed by (6.2).

Clearly, the behavior of the second differential equation, namely (6.7)-(2), would depend on the details of the function $F(q)$. Relevant issues regarding this matter will be analyzed in detail later.

6.1.2 General Solution

Find general solution of the following Clairaut equation.

$$\begin{aligned} (AA) : \quad y &= x \, \frac{\mathrm{d}y}{\mathrm{d}x} + \left(\frac{a}{\frac{\mathrm{d}y}{\mathrm{d}x}} \right) \\ &= x \, q + \left(\frac{a}{q} \right) . \end{aligned} \tag{6.9}$$

According to the procedure explained above, general solution of a Clairaut equation is readily obtained by replacing the variable q—that is, $\frac{\mathrm{d}y}{\mathrm{d}x}$—with parameter σ. Thus, general solution of (AA) is

$$y = x \, \sigma + \left(\frac{a}{\sigma} \right) \; . \tag{6.10}$$

Comment

Equation (6.10) describes a series of straight lines that could be assumed to be parallel to tangents to the parabola

$$y^2 = 4\,a\,x \ . \tag{6.11}$$

If this assumption turns out to be correct, these straight lines and tangents to the parabola will have the same slope—meaning, at any given point (x, y) the value of the differential $\frac{\mathrm{d}y}{\mathrm{d}x}$ derived from (6.11) will equal that obtained from the general solution (6.10).

In order to check the above assumption, proceed as follows. Square root of the equation of the parabola, (6.11), gives $[y]_{parabola} = \sqrt{4\,a\,x}$. Therefore

$$\left[\frac{\mathrm{d}y}{\mathrm{d}x}\right]_{parabola} = \frac{\mathrm{d}\sqrt{4ax}}{\mathrm{d}x} = \sqrt{\left(\frac{a}{x}\right)} \ . \tag{6.12}$$

Now, replace $\frac{\mathrm{d}y}{\mathrm{d}x}$ in Clairaut equation (AA) by $\left[\frac{\mathrm{d}y}{\mathrm{d}x}\right]_{parabola}$—or equivalently replace the existing σ in the solution (6.10) by $\left[\frac{\mathrm{d}y}{\mathrm{d}x}\right]_{parabola}$. Either way, one gets

$$\begin{aligned} y &= x\sqrt{\left(\frac{a}{x}\right)} + \left(\frac{a}{\sqrt{\left(\frac{a}{x}\right)}}\right) \\ &= 2\sqrt{a}\sqrt{x} \ . \end{aligned} \tag{6.13}$$

And this is (the square root of) the equation of the parabola itself!

6.1.3 Singular Solution

Equation (6.13) is clearly a singular solution of the differential equation in example (AA) because while it legitimately solves the differential equation, it is not obtained by tweaking its general solution.

The success of the above exercise is due to fortunate geometrical intuition. Clearly, successful intuition cannot be guaranteed. Therefore, it is important to devise an algebraic procedure for determining singular solutions of Clairaut equations. Such procedure is given below.

An Informal Procedure

(1) Begin with general solution $y \equiv y(x)$.
For instance, general solution for differential equation (AA) is (6.10).

(2) Differentiate general solution, y, with respect to σ. Set the result for $\frac{\mathrm{d}y}{\mathrm{d}\sigma}$ equal to zero. Solve the resultant equation for σ, and notate the solution σ_0. For instance, for the given general solution (6.10), the differential is $\frac{\mathrm{d}y}{\mathrm{d}\sigma} = x - \frac{a}{\sigma^2}$. Now, set it equal to zero

$$\frac{\mathrm{d}y}{\mathrm{d}\sigma} = x - \frac{a}{\sigma^2} = 0 \tag{6.14}$$

and determine the solution for σ The result is $\sigma = \sqrt{\frac{a}{x}}$. Next, notate this σ σ_0. In this manner, one has $\sigma_0 = \sqrt{\frac{a}{x}}$.

(3) Replace σ by σ_0 in the general solution.
For the present case that means

$$\begin{aligned} y &= x\,\sigma_0 + \left(\frac{a}{\sigma_0}\right) \\ &= x\sqrt{\left(\frac{a}{x}\right)} + \left(\frac{a}{\sqrt{\frac{a}{x}}}\right) \\ &= 2\sqrt{ax}\,. \end{aligned} \tag{6.15}$$

Voila! The singular solution.

Squaring both sides of (6.15) leads to the parabola $y^2 = 4ax$.

Formal Procedure

For the Clairaut equation (6.3), the formally suggested second solution—noted in (6.7)-(2)—is most likely the singular solution. It is helpful to determine whether this likelihood holds true.

Comparing (6.3), (6.5), (6.7)-(2) and (6.9), one has $f(q) = \left(\frac{a}{q}\right)$ and $\mathrm{d}f(q) = -\left(\frac{a}{q^2}\right)\mathrm{d}q$. Accordingly, $F(q) = -\left(\frac{a}{q^2}\right)$ and the suggested singular solution, $[x + F(q)] = 0$, that is

$$x - \left(\frac{a}{q^2}\right) = 0\,,$$

yields $q = \sqrt{\frac{a}{x}}$. Integration leads to

$$\begin{aligned} \int q\,\mathrm{d}x &= \int \frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = y \\ = \int \sqrt{\left(\frac{a}{x}\right)}\,\mathrm{d}x &= 2\sqrt{a\,x}\,. \end{aligned}$$

And, as suspected, this is identical to the result in (6.15)—the solution obtained by using the informal procedure! Thus either of the two, formal or informal, procedures for determining the singular solution may be used.

6.1.4 Equations I(B)–I(F)

Find both general and singular solution to the following Clairaut equations numbered I(B)–I(F).

$$\begin{aligned} I(B): &\qquad y = x\,q + 2q^2. \\ I(C): &\qquad y = x\,q + 4q^3. \\ I(D): &\qquad y = x\,q - 3\sin(q). \\ I(E): &\qquad y = x\,q + 5\cos(q). \\ I(F): &\qquad y = x\,q - \sin^2(q). \end{aligned} \tag{6.16}$$

6.1.5 Informal Solution

$$I(B):\ \textit{General Solution is}\ :\qquad y = x\,\sigma + 2\sigma^2\,.$$
$$\textit{Setting,}\ \frac{\mathrm{d}y}{\mathrm{d}\sigma} = x + 4\,\sigma = 0\ ,\ \textit{gives}\ \sigma = \sigma_0 = -\left(\frac{x}{4}\right).$$
$$\textit{Singular solution is}\ :\ y = x\left(\frac{-x}{4}\right) + 2\left(\frac{-x}{4}\right)^2 = -\left(\frac{x^2}{8}\right).$$

$$I(C):\ \textit{General Solution is}\ :\qquad y = x\,\sigma + 4\sigma^3\ .$$
$$\textit{Setting}\ \ \frac{\mathrm{d}y}{\mathrm{d}\sigma} = x + 12\,\sigma^2 = 0\ \ \textit{gives}\ \sigma_0 = \sqrt{-\frac{x}{12}}\ .$$
$$\textit{Accordingly, singular solution is}\ :\ y = \sqrt{-\frac{x}{12}}\left[x + 4\left(\sqrt{-\frac{x}{12}}\right)^2\right].$$
$$\textit{Equivalently}\ ,\ y = \sqrt{-\frac{x}{12}}\left[\frac{2x}{3}\right]\ ;\ \textit{Or}\ ,\ 27y^2 + x^3 = 0\ .$$

$$I(D):\ \textit{General Solution is}\ :\qquad y = x\,\sigma - 3\sin(\sigma)$$
$$\textit{Setting}\ \ \frac{\mathrm{d}y}{\mathrm{d}\sigma} = x - 3\cos(\sigma) = 0\ ,\ \textit{gives}\ \sigma_0 = \cos^{-1}\left(\frac{x}{3}\right).$$
$$\textit{Singular solution is}\ :\ y = x\cos^{-1}\left(\frac{x}{3}\right) - 3\sin\left[\cos^{-1}\left(\frac{x}{3}\right)\right].$$

$$I(E): \ General\ Solution\ is\ : \qquad y = x\,\sigma + 5\cos(\sigma).$$
$$Setting\ \frac{\mathrm{d}y}{\mathrm{d}\sigma} = x - 5\,\sin(\sigma) = 0\ ,\ gives\ \sigma_0 = \sin^{-1}\left(\frac{x}{5}\right).$$
$$Accordingly,\ singular\ solution\ is\ :\ y - x\sin^{-1}\left(\frac{x}{5}\right) = 5\cos(\sigma_0)\ .$$

$$I(F): \ General\ Solution\ is\ : \qquad y = x\,\sigma - \sin^2(\sigma)\,.$$
$$Setting\ \frac{\mathrm{d}y}{\mathrm{d}\sigma} = x - \sin(2\,\sigma) = 0\ ,\ gives\ \sigma_0 = \left(\frac{1}{2}\right)\sin^{-1}(x)\,.$$
$$Singular\ solution\ is\ :\ y = \left(\frac{x}{2}\right)\sin^{-1}(x) - \sin^2\left[\frac{1}{2}\sin^{-1}(x)\right]. \qquad (6.17)$$

6.1.6 Formal Solution

$$F(q) = \frac{\mathrm{d}(2\,q^2)}{\mathrm{d}q};$$
$$Equation\ I(B): \ F(q) + x = 4q + x = 0\,.$$
$$Therefore\ \ 4\int\frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = 4y = -\int(x)\,\mathrm{d}x = -\frac{x^2}{2} + const$$
$$Singular\ solution\ is\ :\ y = -\left(\frac{x^2}{8}\right) + constant\ .$$

$$F(q) = \frac{\mathrm{d}(4\,q^3)}{\mathrm{d}q};$$
$$Equation\ I(C): \ F(q) + x = 12\,q^2 + x = 0\,.$$
$$Therefore \qquad \int q\mathrm{d}x = \int\mathrm{d}y = \sqrt{\frac{-1}{12}}\int\sqrt{x}\,\mathrm{d}x.$$
$$Singular\ solution\ is\ :\ y = \sqrt{\frac{-1}{12}}\left(\frac{2}{3}\right)x^{\frac{3}{2}} + const.$$

$$F(q) = \frac{\mathrm{d}[-3\,\sin(q)]}{\mathrm{d}q}\ ;$$
$$Equation\ I(D): \ F(q) + x = -3\,\cos(q) + x = 0\,.$$
$$Therefore \qquad \int q\mathrm{d}x = \int\mathrm{d}y = \int\cos^{-1}\left(\frac{x}{3}\right)\mathrm{d}x.$$
$$Singular\ solution\ is\ :\ y = x\,\cos^{-1}\left(\frac{x}{3}\right) - \sqrt{9 - x^2} + const\ .$$

$$Equivalently\ ,\ y\ =\ x\ \cos^{-1}\left(\frac{x}{3}\right)-\ 3\ \sin\left[\cos^{-1}\left(\frac{x}{3}\right)\right]+const.$$

$$F(q)\ =\ \frac{\mathrm{d}[5\ \cos(q)]}{\mathrm{d}q};$$

$$Equation\ \ I(E):\ \ F(q)\ +\ x\ =\ -\ 5\ \sin(q)\ +\ x\ =\ 0\,.$$

$$Therefore\ \ \int \mathrm{d}y\ =\ \int \sin^{-1}\left(\frac{x}{5}\right)\mathrm{d}x\,.$$

$$Singular\ solution\ is\ \ :\ \ y\ =\ x\ \sin^{-1}\left(\frac{x}{5}\right)\ +\ \sqrt{25-x^2}+const.$$

$$Equivalently\ ,\ y\ =\ x\ \sin^{-1}\left(\frac{x}{5}\right)+\ 5\ \cos\left[\sin^{-1}\left(\frac{x}{5}\right)\right]+const.$$

$$F(q)\ =\ \frac{\mathrm{d}[-\ \sin^2(q)]}{\mathrm{d}q};$$

$$Equation\ \ I(F):\ \ F(q)\ +\ x\ =\ -\ \sin(2\,q)\ +\ x\ =\ 0$$

$$Therefore\ \ \int \mathrm{d}y\ =\ \left(\frac{1}{2}\right)\int \sin^{-1}(x)\,\mathrm{d}x$$

$$Singular\ solution\ is\ \ :\ \ y\ =\ \left(\frac{x}{2}\right)\ \sin^{-1}(x)+\left(\frac{1}{2}\right)\sqrt{1-x^2}+const$$

$$Equivalently\ ,\ y\ =\ \left(\frac{x}{2}\right)\ \sin^{-1}(x)-\ \sin^2\left[\left(\frac{1}{2}\right)\sin^{-1}(x)\right]+const.$$

(6.18)

6.1.7 Problems Group I

Solve the following Clairaut differential equations problems labeled (1)–(4). Find both the general solution and the singular solution. [Hint: Read (6.10)–(6.18)]

$$y\ =\ x\,q+3\,q^2\ .\ (1)$$
$$y\ =\ x\,q+3\,q^3\ .\ (2)$$
$$y\ =\ x\,q-2\ \sin q\ .\ (3)$$
$$y\ =\ x\,q+2\ \cos q\ .\ (4)$$

6.2 Lagrange Equation

Similar to the Clairaut differential equation, but somewhat more general, is a Lagrange equation that can be written as

$$y \;=\; x\,G(q) \;+\; F(q) \;\;. \tag{6.19}$$

If the function $G(q)$ should be equal to q, Lagrange equation (6.19) would reduce to Clairaut equation.

6.2.1 General Solution

Differentiate (6.19)

$$\mathrm{d}y \;=\; x\,\mathrm{d}G(q) \;+\; G(q)\,\mathrm{d}x \;+\; \mathrm{d}F(q)$$

and use the following relationship proposed in (6.1)

$$\mathrm{d}y \;=\; q\,\mathrm{d}x \;.$$

The result is

$$q\,\mathrm{d}x \;=\; x\,\mathrm{d}G(q) \;+\; G(q)\,\mathrm{d}x \;+\; \mathrm{d}F(q) \quad .$$

Now move $G(q)\,\mathrm{d}x$ to the left-hand side

$$[q \;-\; G(q)]\,\mathrm{d}x \;=\; x\,\mathrm{d}G(q) + \mathrm{d}F(q)$$

and divide both sides by $\mathrm{d}q$

$$[q \;-\; G(q)]\,\frac{\mathrm{d}x}{\mathrm{d}q} \;=\; x\,\frac{\mathrm{d}G(q)}{\mathrm{d}q} + \frac{\mathrm{d}F(q)}{\mathrm{d}q} \quad .$$

Assume $q \;\neq\; G(q)$. Multiply from the left both sides by $\frac{1}{[q \;-\; G(q)]}$.

$$\frac{\mathrm{d}x}{\mathrm{d}q} \;=\; \frac{1}{[q \;-\; G(q)]}\left[x\,\frac{\mathrm{d}G(q)}{\mathrm{d}q} + \frac{\mathrm{d}F(q)}{\mathrm{d}q}\right] \;\;, \tag{6.20}$$

This is a first-order differential equation, and its solution, $x(q)$, should be achievable depending on the details of the functions $G(q)$ and $F(q)$. Even its singular solution may be found be setting the singular value $q = G(q)$ in the original differential equation.

6.2.2 Examples II(A)–II(C)

$$
\begin{aligned}
II(A): \quad & y = x\left(\frac{3q}{2}\right) - q^2 \ . \\
II(B): \quad & y = x\left(\frac{q}{3}\right) - \left(\frac{q^2}{6}\right) \ . \\
II(C): \quad & y = 3\,q\,x + 5\,\log(q) \ .
\end{aligned}
\tag{6.21}
$$

Solution: II(A)

Here, $G(q) = \left(\frac{3q}{2}\right)$ and $F(q) = -q^2$. Therefore, (6.20) and II(A) give

$$
\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}q} &= \frac{1}{\left(\frac{-q}{2}\right)}\left[x\left(\frac{3}{2}\right) - 2\,q\right] \\
&= -\left(\frac{3}{q}\right)x + 4 \ .
\end{aligned}
\tag{6.22}
$$

This is a first-order differential equation. Its solution $x(q)$ can be found by using (4.10) and (4.15). The result is

$$
x(q) = q + \frac{\sigma_1}{q^3} \ , \tag{6.23}
$$

where σ_1 is a constant. Inserting (6.23) into Lagrange equation II(A) leads to

$$
y(q) = \frac{1}{2}\left[q^2 + \sigma_1\left(\frac{3}{q^2}\right)\right] \ . \tag{6.24}
$$

Equations (6.23) and (6.24) are parametric representation of the general solution $y = y(x)$.

Singular solution is obtained by setting $q = G(q)$. That means using the relationship $q = 3\frac{q}{2}$, which is $q = 0$, in equation II(A). Thus, the singular solution is

$$
y(x) = 0 \ . \tag{6.25}
$$

Solution: II(B)

Here, $G(q) = \left(\frac{q}{3}\right)$ and $F(q) = -\left(\frac{q^2}{6}\right)$. Therefore, (6.20) and II(B) give

$$
\frac{\mathrm{d}x}{\mathrm{d}q} = \frac{1}{\left(\frac{2q}{3}\right)}\left[\left(\frac{x}{3}\right) - \left(\frac{q}{3}\right)\right]
$$

$$= \left(\frac{1}{2}\right)\left[\left(\frac{x}{q}\right) - 1\right] \quad . \tag{6.26}$$

This is a first-order differential equation. Its solution, according to (4.10) and (4.15), is

$$x(q) = -q \; - \sigma_1 \sqrt{q} \quad . \tag{6.27}$$

As usual, σ_1 is an arbitrary constant. Inserting (6.27) into Lagrange equation II(B) leads to

$$y(q) = -\frac{1}{3}\left[\frac{3}{2}q^2 \; + \; \sigma_1 q^{\frac{3}{2}}\right] \quad . \tag{6.28}$$

Equations (6.27) and (6.28) are parametric equations representing the general solution $y = y(x)$. Singular solution is obtained in similar fashion to that for equation II(A). And the result once again is $y(x) \; = \; 0$.

Solution: II(C)

Here, $G(q) \; = (3\,q)$ and $F(q) \; = \; 5\log(q)$. Therefore, (6.20) and II(C) give

$$\frac{\mathrm{d}x}{\mathrm{d}q} = -\left(\frac{1}{2q}\right)\left[3x + \left(\frac{5}{q}\right)\right] \quad . \tag{6.29}$$

This is a first-order differential equation. Its solution, according to (4.10) and (4.15), is

$$x(q) = -\left(\frac{5}{q}\right) + \left[\frac{\sigma_1}{q^{\frac{3}{2}}}\right] \quad . \tag{6.30}$$

As usual, σ_1 is an arbitrary constant. Inserting (6.30) into Lagrange equation II(C) leads to

$$y(q) = 5[\log(q) - 3] + 3\left[\frac{\sigma_1}{q^{\frac{1}{2}}}\right] \quad . \tag{6.31}$$

Equations (6.30) and (6.31) are parametric equations representing the general solution $y = y(x)$. Singular solution is obtained in similar fashion to that for equation II(A). And the result once again is $y(x) \; = \; 0$.

6.2.3 Problems Group II

Solve the following Lagrange differential equations problems labeled (1)–(3). Find both the general solution and the singular solution. [Hint: Read (6.21)–(6.31).]

$$\begin{aligned}
(1): \quad & y = \left(\frac{q}{3}\right) x - 3q^2 \ . \\
(2): \quad & y = \left(\frac{q}{5}\right) x - \left(\frac{q^2}{5}\right) \ . \\
(3): \quad & y = 4q\,x + 4\,\log(q) \ .
\end{aligned}$$

6.3 Separable Differential Equations

Whenever it is possible to separate the dependent and the independent variables and express the differential equation in the form

$$\frac{\mathrm{d}y}{Q(y)} = \frac{\mathrm{d}x}{P(x)} \tag{6.32}$$

the resultant equation

$$\int \frac{\mathrm{d}y}{Q(y)} = \int \frac{\mathrm{d}x}{P(x)} \tag{6.33}$$

can directly be integrated.

6.3.1 Examples Group III

Variables are separated in the following four problems, and then, direct integration is used.

$$\begin{aligned}
(1): \quad & \frac{\mathrm{d}y}{y\,\log y} = \exp(x)\,\mathrm{d}x \ . \\
(2): \quad & \frac{\mathrm{d}y}{\mathrm{d}x} = \exp(2\,x)\ \exp(-2\,y) \ . \\
(3): \quad & (\sin x)\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = (\cos x)\ \frac{\mathrm{d}y}{\mathrm{d}x} \ . \\
(4): \quad & \frac{\mathrm{d}^3y}{\mathrm{d}x^3} = (x^2 + 2x)\ \exp x \ .
\end{aligned} \tag{6.34}$$

6.3.2 Solution

$$(1): \int \left[\frac{\mathrm{d}y}{y\,\log y}\right] = \int \exp(x)\,\mathrm{d}x\;;\quad \log[\log(y)] = \exp(x) + const\;.$$
$$Equivalently\;:\; y = \log^{-1}\log^{-1}\left[\exp(x) + \sigma_0\right]\;.$$

$$(2): \int \frac{\mathrm{d}y}{\exp(-2\,y)} = \int \exp(2\,x)\,\mathrm{d}x\;;\quad \frac{\exp(2\,y)}{2} = \frac{\exp(2\,x)}{2} + const\;.$$
$$Equivalently\;: y = \frac{1}{2}\log\left[\sigma_0 + \exp(2\,x)\right].$$

$$(3): \left[Use\;\frac{\mathrm{d}y}{\mathrm{d}x} = q\right];\; (\sin x)\,\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = (\sin x)\,\frac{\mathrm{d}q}{\mathrm{d}x} = (\cos x)\frac{\mathrm{d}y}{\mathrm{d}x} = (\cos x)\,q$$
$$Therefore,\; \int \frac{\mathrm{d}q}{q} = \int \cot x\,\mathrm{d}x\;;\; \log q = \log(\sin x) + const\;;$$
$$Equivalently\;,\; q \equiv \frac{\mathrm{d}y}{\mathrm{d}x} = \sigma_0\,\sin x\;.$$
$$Or\;,\; \int \mathrm{d}y = \sigma_0 \int \sin x\,\mathrm{d}x\;;\; y = -\sigma_0\,\cos x + \sigma_1\;.$$

$$(4): \int \left(\frac{\mathrm{d}^3y}{\mathrm{d}x^3}\right)\mathrm{d}x = \left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right) = \int (x^2 + 2x)\,\exp(x)\,\mathrm{d}x = x^2\,\exp x + \sigma_0$$
$$Therefore\;,\; \int \left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)\mathrm{d}x = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \int x^2.\,\exp(x)\,\mathrm{d}x + \sigma_0 x$$
$$= (x^2 - 2x + 2)\,\exp x + \sigma_0 x + \sigma_1\;.$$
$$Thus\;,\; \int \mathrm{d}y = \int [(x^2 - 2x + 2)\,\exp x + \sigma_0 x + \sigma_1]\,\mathrm{d}x\;.$$
$$Or\;,\; y = (x^2 - 4x + 6)\,\exp x + \frac{\sigma_0}{2}x^2 + \sigma_1\,x + \sigma_2\;. \qquad (6.35)$$

6.3.3 Problems Group III

Solve the following problems by separating the variables.

$$(1): \qquad \frac{\mathrm{d}y}{y} = \frac{\mathrm{d}x}{x}\;.$$

$$(2): \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \exp(3y - 2x) \ .$$

$$(3): \quad \frac{\mathrm{d}y}{\mathrm{d}x} = 2\,x\,y\,\exp(x^2) \ .$$

$$(4): \quad \frac{\mathrm{d}y}{\mathrm{d}x} = x^2 y^2 \ .$$

6.4 Separable Equations of Form $\frac{\mathrm{d}y}{\mathrm{d}x} = \Phi\left(\frac{y}{x}\right)$

Another readily separable class of equations are of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \Phi\left(\frac{y}{x}\right) \ . \tag{6.36}$$

6.4.1 Solution $\frac{\mathrm{d}y}{\mathrm{d}x} = \Phi\left(\frac{y}{x}\right)$

Introduce a new variable

$$\rho = \left(\frac{y}{x}\right) . \tag{6.37}$$

As a result, $y = x\,\rho$ and (6.36) can be represented as

$$\Phi(\rho) = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}(x\rho)}{\mathrm{d}x} = x\,\frac{\mathrm{d}\rho}{\mathrm{d}x} + \rho \ . \tag{6.38}$$

Transferring ρ to the left-hand side,

$$\Phi(\rho) - \rho = x\,\frac{\mathrm{d}\rho}{\mathrm{d}x} \ ,$$

and multiplying from the left both sides by $\frac{\mathrm{d}x}{x}\left(\frac{1}{\Phi(\rho)-\rho}\right)$ finally transforms (6.36) into one of separable form

$$\frac{\mathrm{d}x}{x} = \left[\frac{\mathrm{d}\rho}{\Phi(\rho) - \rho}\right] \ . \tag{6.39}$$

Equation (6.39), depending on complexity of $\Phi(\rho)$, may now be integrated.

6.4.2 Examples Group IV: $\frac{dy}{dx} = \Phi\left(\frac{y}{x}\right)$ Equations

Examples Group IV: Equations (1)–(6)

$$
\begin{aligned}
(1): \quad & \frac{dy}{dx} = 1 + \left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2 \ . \\
(2): \quad & \frac{dy}{dx} = \tan\left[\frac{y}{x}\right] + \left(\frac{y}{x}\right) \ . \\
(3): \quad & \frac{dy}{dx} = \exp\left[-\left(\frac{y}{x}\right)\right] + \left(\frac{y}{x}\right) \ . \\
(4): \quad & \frac{dy}{dx} = \left(\frac{x}{y}\right) + \left(\frac{y}{x}\right) \ . \\
(5): \quad & x\,(x^2 + y^2)\frac{dy}{dx} = y\,(x + y)^2 \, . \\
(6): \quad & \frac{dy}{dx} = \left(\frac{y + cx}{x + cy}\right) \ .
\end{aligned}
\tag{6.40}
$$

6.4.3 Solution

As noted in (6.38) and (6.37), substitute $\Phi(\rho)$ for $\frac{dy}{dx}$ and ρ for $\left(\frac{y}{x}\right)$. One gets

$$
\begin{aligned}
(1): \ \Phi(\rho) &= 1 + \rho + \rho^2 \ . \ Therefore, \ according \\
\int \frac{dx}{x} &= \left[\frac{d\rho}{\Phi(\rho) - \rho}\right] = \int \left[\frac{d\rho}{1 + \rho^2}\right] , or \\
\left[\sigma_0 + \log x\right] &= \tan^{-1}\rho \ , \ or \ \rho = \tan\left[\sigma_0 + \log x\right] , \ or \\
y &= x \tan\left[\sigma_0 + \log x\right] \ .
\end{aligned}
\tag{6.39}
$$

$$
\begin{aligned}
(2): \ \Phi(\rho) &= \tan\rho + \rho \ . \ Therefore, \quad \int \left[\frac{d\rho}{\tan\rho}\right] dx = \int \frac{dx}{x} , \ or \\
\log(\sin\rho) &= \log\sigma_0 + \log x \ , \ \sin\rho = \sigma_0 x \ , \ or \\
y &= x \sin^{-1}\left[\sigma_0 x\right] \ .
\end{aligned}
$$

$$
\begin{aligned}
(3): \ \Phi(\rho) &= \exp(-\rho) + \rho \ . \ Therefore, \quad \int \left[\frac{d\rho}{\exp(-\rho)}\right] = \int \frac{dx}{x} , \ or \\
\exp(\rho) &= \sigma_0 + \log x; \rho = \log\left[\sigma_0 + \log x\right] , \ or \ y = x \log\left[\sigma_0 + \log x\right] \ .
\end{aligned}
$$

$$(4): \quad \Phi(\rho) = \frac{1}{\rho} + \rho \ . \ Therefore, \quad \int \left[\frac{\mathrm{d}\rho}{\frac{1}{\rho}}\right] = \int \frac{\mathrm{d}x}{x} \ , \ or$$

$$\frac{\rho^2}{2} = \sigma_0 + \log x \ , \ or \ y = \pm x\sqrt{2}\left[\sigma_0 + \log x\right]^{\frac{1}{2}} \quad .$$

$$(5) \ Write \ as: \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{y}{x}\right)\left[1 + \frac{2\left(\frac{y}{x}\right)}{1 + \left(\frac{y}{x}\right)^2}\right] . Therefore \ ,$$

$$\Phi(\rho) = \frac{2\rho^2}{1+\rho^2} + \rho \ . \ Accordingly, \quad \int \left[\frac{(1+\rho^2)}{2\rho^2}\right]\mathrm{d}\rho = \int \frac{\mathrm{d}x}{x}, \ or$$

$$-\frac{1}{2\rho} + \frac{\rho}{2} = \sigma_0 + \log x \ . \ Thus: \ \rho^2 - 2\rho\,(\sigma_0 + \log x) - 1 = 0 \ , \ or$$

$$\rho = (\sigma_0 + \log x) \pm \sqrt{(\sigma_0 + \log x)^2 + 1} \ , or$$

$$y = x\,(\sigma_0 + \log x) \pm x\sqrt{(\sigma_0 + \log x)^2 + 1}.$$

$$(6): \ \Phi(\rho) = (\rho + c)/(1 + c\rho) \ . \ Therefore,$$

$$\left[\frac{\mathrm{d}\rho}{\Phi(\rho) - \rho}\right] = \int \left[\frac{\mathrm{d}\rho}{(\rho + c)/(1 + c\rho) - \rho}\right] = \frac{1}{c}\int \left[\frac{1 + c\rho}{1 - \rho^2}\right]\mathrm{d}\rho$$

$$= (2c)^{-1}\left[-(c-1)\int \frac{\mathrm{d}\rho}{1+\rho} + (c+1)\int \frac{\mathrm{d}\rho}{1-\rho}\right] = \int \frac{\mathrm{d}x}{x}$$

$$= -(2c)^{-1}\left[(c-1)\log(1+\rho) + (c+1)\log(1-\rho)\right] = \left[\sigma_0 + \log x\right]. \quad (6.41)$$

6.4.4 Problems Group IV

Solve the following $\frac{\mathrm{d}y}{\mathrm{d}x} = \Phi\left(\frac{y}{x}\right)$ problems . [Hint: Read (6.40) and (6.41).]

$$(1): \quad \frac{\mathrm{d}y}{\mathrm{d}x} = 2\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2 \quad .$$

$$(2): \quad \frac{\mathrm{d}y}{\mathrm{d}x} = 3\,\tan\left[\frac{y}{x}\right] + \left(\frac{y}{x}\right) \quad .$$

$$(3): \quad \frac{\mathrm{d}y}{\mathrm{d}x} = 5\,\exp\left[-3\left(\frac{y}{x}\right)\right] + \left(\frac{y}{x}\right) \quad .$$

$$(4): \quad \frac{\mathrm{d}y}{\mathrm{d}x} = 4\left(\frac{x}{y}\right) + \left(\frac{y}{x}\right) \quad .$$

$$(5): \ x\,(x^2 + 2\,y^2)\frac{\mathrm{d}y}{\mathrm{d}x} = y\,\left(x^2 + 3\,y\,x + 2\,y^2\right) \quad .$$

6.5 Equations Reducible to $\frac{dy}{dx} = \Phi\left(\frac{y}{x}\right)$

These are equations of the form

$$\frac{dy}{dx} = \phi\left[\frac{b_1 y + c_1}{b_2 x + c_2}\right] .$$

6.5.1 Examples Group V: (I)–(IV)

Solved below are equations (I) to (IV) that are of the form described above. Despite slightly different appearance, they are readily reducible to the general equation $\frac{dy}{dx} = \Phi\left(\frac{y}{x}\right)$.

$$(I): \quad \frac{dy}{dx} = 1 + \left(\frac{y}{x+2}\right) + \left(\frac{y}{x+2}\right)^2 .$$

$$(II): \quad \frac{dy}{dx} = 2\left[\frac{y+1}{x+2}\right] + 4 .$$

$$(III): \quad \frac{dy}{dx} = \tan\left[\frac{y+2}{x+3}\right] + \left[\frac{y+2}{x+3}\right] .$$

$$(IV): \quad \frac{dy}{dx} = \exp\left[-\left(\frac{y+3}{x+4}\right)\right] + \left(\frac{y+3}{x+4}\right) .$$

Solution: (I)

$$Set \;\; (x+2) = z \;\; . \;\; Then \;\; \frac{dz}{dx} = 1 \;\; and \;\; \frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{dy}{dz} \cdot 1$$

$$Therefore , \;\; \frac{dy}{dz} = \frac{dy}{dx} = 1 + \left(\frac{y}{z}\right) + \left(\frac{y}{z}\right)^2 . \tag{6.42}$$

Except for change of variable from $x \to z$, this is identical to (1) in (6.40). Therefore, its solution is

$$(I): \quad y = z\tan\left[\sigma_0 + \log z\right] \;\; , \;\; or \;\; , \;\; y = (x+2)\tan\left[\sigma_0 + \log(x+2)\right] . \tag{6.43}$$

Solutions: (II)–(IV)

Set $z_1 = y + 1$, $z_2 = x + 2$ and $\left(\frac{z_1}{z_2}\right) = \rho$. Then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}z_1}{\mathrm{d}z_2} .$$

Also, rewriting equation (II), one has

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\left(\frac{z_1}{z_2}\right) + 4 = 2\rho + 4.$$

If one sets

$$2\rho + 4 = \Phi(\rho) ,$$

then, much like (6.38), one has $\frac{\mathrm{d}y}{\mathrm{d}x} = \Phi(\rho)$. Therefore, according to (6.39), $\frac{\mathrm{d}x}{x} = \left[\frac{\mathrm{d}\rho}{\Phi(\rho)-\rho}\right]$. Additionally, because $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}z_1}{\mathrm{d}z_2}$, the earlier variable y corresponds to the current variable z_1. Similarly, the earlier variable x corresponds to the current variable z_2. Therefore, for brevity it is convenient to go directly to (6.39) and rewrite it by transliterating x to z_2. Additionally, we do the same for the corresponding result that would be obtained.

$$\frac{\mathrm{d}x}{x} \rightarrow \frac{\mathrm{d}z_2}{z_2} = \left[\frac{\mathrm{d}\rho}{\Phi(\rho)-\rho}\right] = \left[\frac{\mathrm{d}\rho}{2\rho+4-\rho}\right] = \left[\frac{\mathrm{d}\rho}{\rho+4}\right]. \tag{6.44}$$

Integrating both sides leads to

$$\begin{aligned} \int \frac{\mathrm{d}z_2}{z_2} &= \int \left[\frac{\mathrm{d}\rho}{\rho+4}\right] , \\ \log z_2 + constant &= \log(\rho+4) , \\ \sigma_0\, z_2 &= \rho + 4 . \end{aligned} \tag{6.45}$$

Using the information, $\rho = \frac{z_1}{z_2}$, $z_1 = y+1$, $z_2 = x+2$, the above relationship becomes $\sigma_0(x+2) = \left(\frac{y+1}{x+2}\right) + 4$ which leads to the result

$$(II): \qquad y = \sigma_0(x+2)^2 - 4x - 9 . \tag{6.46}$$

$$(III): \qquad Set \;\; z_1 = y+2 \;\; , \;\; z_2 = x+3 \;\; , \;\; \left(\frac{z_1}{z_2}\right) = \rho \;\; .$$

$$Then \;\; , \;\; \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}z_1}{\mathrm{d}z_2} = \tan\left(\frac{z_1}{z_2}\right) + \left(\frac{z_1}{z_2}\right) = \tan\rho + \rho \;\; . \tag{6.47}$$

As per (6.38) and (6.39), the relationship $\frac{\mathrm{d}x}{x} = \left[\frac{\mathrm{d}\rho}{\Phi(\rho)-\rho}\right]$ obtains. And again because here $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \left(\frac{\mathrm{d}z_1}{\mathrm{d}z_2}\right)$, therefore, similar to (6.39), one can write

$$\frac{dz_2}{z_2} = \left[\frac{d\rho}{\Phi(\rho) - \rho}\right] = \left[\frac{d\rho}{\tan\rho + \rho - \rho}\right] = \left[\frac{d\rho}{\tan\rho}\right]. \tag{6.48}$$

Integrating both sides leads to

$$\int \frac{dz_2}{z_2} = \int \left[\frac{d\rho}{\tan\rho}\right],$$

$$\log z_2 + constant = \log(\sin\rho), \tag{6.49}$$

$$\sigma_0 z_2 = \sin\rho \;;\; \rho = \sin^{-1}(\sigma_0 z_2). \tag{6.50}$$

Using the information, $z_1 = y + 2$, $z_2 = x + 3$, $\rho = \frac{z_1}{z_2}$, the above relationship becomes $\left(\frac{y+2}{x+3}\right) = \sin^{-1}[\sigma_0 (x+3)]$ which leads to the result

$$(III): \qquad y = (x+3)\sin^{-1}[\sigma_0 (x+3)] - 2. \tag{6.51}$$

$$(IV): \qquad Set \;\; z_1 = y + 3 \;,\; z_2 = x + 4 \;,\; \left(\frac{z_1}{z_2}\right) = \rho \;.$$

$$Then \;,\; \frac{dy}{dx} = \frac{dz_1}{dz_2} = \exp\left(-\frac{z_1}{z_2}\right) + \left(\frac{z_1}{z_2}\right) = \exp(-\rho) + \rho \;. \tag{6.52}$$

Following the same procedure as used in the above two equations, namely (II) and (III), one can write

$$\frac{dz_2}{z_2} = \left[\frac{d\rho}{\Phi(\rho) - \rho}\right] = \left[\frac{d\rho}{\exp(-\rho) + \rho - \rho}\right] = \left[\frac{d\rho}{\exp(-\rho)}\right]. \tag{6.53}$$

Integrating both sides leads to

$$\int \frac{dz_2}{z_2} = \int \left[\frac{d\rho}{\exp(-\rho)}\right],$$

$$\log(z_2) + \sigma_0 = \exp(\rho),$$

$$\rho = \log\left[\log(z_2) + \sigma_0\right] \;. \tag{6.54}$$

Because $z_1 = y + 3$, $z_2 = x + 4$, $\rho = \frac{z_1}{z_2}$, (6.54) readily leads to the result

$$(IV): \qquad y = (x+4)\log\left[\log(x+4) + \sigma_0\right] - 3. \tag{6.55}$$

6.5.2 Equations $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$

$$
\begin{aligned}
(V): \qquad & \frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{x-y}{x+y+2}\right). \\
(VI): \qquad & \frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{x-y+1}{x+y+3}\right). \\
(VII): \qquad & \frac{\mathrm{d}y}{\mathrm{d}x} = \left[\frac{x+2y+3}{4(x+2y)+5}\right].
\end{aligned} \tag{6.56}
$$

Solutions of (V) and (VII)

Unlike (II)–(IV) the situation is different here because either the numerator or the denominator, or both, contain x as well as y. When such is the case, one proceeds as follows: First one sets

$$
y = z_1 + a \;;\; x = z_2 + b \;;\; \left(\frac{z_1}{z_2}\right) = \rho \;;\; \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}z_1}{\mathrm{d}z_2}\,. \tag{6.57}
$$

Then, one eliminates both a and b as well as the original constants in the numerator and the denominator. To that end, one employs Cramer's rule—see (3.117). This procedure is demonstrated in the treatment of (V) and (VI) below.

Equation (V): Solution

$$
(V): \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{x-y}{x+y+2}\right) = \frac{z_2 - z_1 + (-a+b)}{z_2 + z_1 + (a+b+2)}. \tag{6.58}
$$

Both $(-a+b)$ and $(a+b+2)$ are vanishing when $b = a = -1$. As such $z_1 = (y-a) = (y+1)$, $z_2 = (x-b) = (x+1)$, and $\rho = \frac{z_1}{z_2} = \left(\frac{y+1}{x+1}\right)$.

Accordingly, the differential equation (V) becomes

$$
\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}z_1}{\mathrm{d}z_2} = \frac{z_2 - z_1}{z_1 + z_2} = \frac{1-\rho}{1+\rho} = \Phi(\rho)\;. \tag{6.59}
$$

Following the same procedure as in $(II)-(IV)$, one can write

$$
\frac{\mathrm{d}z_2}{z_2} = \left[\frac{\mathrm{d}\rho}{\Phi(\rho) - \rho}\right] = \left[\frac{\mathrm{d}\rho}{\left(\frac{1-\rho}{1+\rho}\right) - \rho}\right] = \frac{(1+\rho)}{1 - 2\rho - \rho^2}\mathrm{d}\rho\,. \tag{6.60}
$$

Integrating both sides leads to

$$\log(z_2) = -\frac{1}{2}\log(\rho^2 + 2\rho - 1) + \sigma_1 \,. \tag{6.61}$$

Replacing z_2 by $(x+1)$ and ρ by $\left(\frac{y+1}{x+1}\right)$ gives

$$\log(x+1) = -\frac{1}{2}\log[y^2 + 4y + 2yx - x^2 + 2] + \log(x+1) + \sigma_1.$$

Thus

$$\log[y^2 + 4y + 2yx - x^2 + 2] = 2\,\sigma_1 \;\; .$$

Exponentiation leads to

$$y^2 + 4y + 2yx - x^2 - \sigma_0 = 0 \;\; .$$

This is a quadratic in y with solutions:

$$(V): \qquad y = -(x+2) \pm \sqrt{2x^2 + 4x + \sigma_0}. \tag{6.62}$$

$$(VI): \qquad \frac{dy}{dx} = \left(\frac{x - y + 1}{x + y + 3}\right) = \frac{z_2 - z_1 + (-a + b + 1)}{z_2 + z_1 + (a + b + 3)}. \tag{6.63}$$

Equation (VI) : Solution

Both $(-a + b + 1)$ and $(a + b + 3)$ are vanishing when $a = -1$ and $b = -2$. As such $z_1 = (y - a) = (y + 1)$, $z_2 = (x - b) = (x + 2)$ and $\rho = \frac{z_1}{z_2} = \left(\frac{y+1}{x+2}\right)$. As a result, the differential equation (VI) becomes

$$\frac{dy}{dx} = \frac{dz_1}{dz_2} = \frac{z_2 - z_1}{z_1 + z_2} = \frac{1 - \rho}{1 + \rho} = \Phi(\rho) \;\; . \tag{6.64}$$

In this form, it becomes identical to (V) above—see (6.59). Therefore, its solution is as obtained in (6.62).

$$\log(z_2) = -\frac{1}{2}\log(\rho^2 + 2\rho - 1) + \sigma_1 \,. \tag{6.65}$$

Replacing z_2 by $(x+2)$ and ρ by $\left(\frac{y+1}{x+2}\right)$ leads to

$$\log(x+2) = -\frac{1}{2}\log[y^2 + 6y + 2yx - x^2 - 2x + 1] + \log(x+2) + \sigma_1.$$

Thus

$$\log[y^2 + 6y + 2yx - x^2 - 2x + 2] = 2\,\sigma_1.$$

Exponentiating the above gives

$$y^2 + 6y + 2yx - x^2 - 2x - \sigma_0 = 0$$

which is a quadratic in y with solutions:

$$(VI): \qquad y = -(x+3) \pm \sqrt{2x^2 + 8x + \sigma_0}. \tag{6.66}$$

Remark

Differential equation (VI) that was solved above had variables $x,\ y$ as well as constants both in the numerator and the denominator. It was successfully treated by replacing x and y with two new variables such that the use of Cramer's rule helped eliminate the constants. There are, however, situations where the Cramer's rule is inapplicable. When that is the case, and the differential equation has features similar to (VII), one proceeds somewhat differently.

Procedure for Solving (VII)

For generality and ease of recognizing the feature of interest, it is convenient to rewrite problem (VII) as

$$\begin{aligned} \mathrm{d}y &= \left[\frac{x + 2y + 3}{4(x + 2y) + 5}\right]\mathrm{d}x \\ &\equiv \left[\frac{(a\,x + b\,y) + c}{j\ (a\,x + b\,y) + r}\right]\mathrm{d}x \end{aligned} \tag{6.67}$$

where $a,\ b,\ c,\ j$, and r are constants. Clearly, the Cramer's rule is inapplicable here because the relevant determinant,

$$\begin{vmatrix} a & b \\ j\,a & j\,b \end{vmatrix}, \tag{6.68}$$

is vanishing.

To deal with this matter, let us introduce a single variable, z_0, for the combination $(ax + by)$.

$$z_0 = a\,x + b\,y\ . \tag{6.69}$$

Differentiate (6.69) and use (6.67) and (6.69).

$$\begin{aligned} \mathrm{d}z_0 &= a\,\mathrm{d}x + b\,\mathrm{d}y \\ &= a\,\mathrm{d}x + b\left[\frac{(a\,x + b\,y) + c}{j\;(a\,x + b\,y) + r}\right]\mathrm{d}x \\ &= a\,\mathrm{d}x + b\left[\frac{z_0 + c}{j\,z_0 + r}\right]\mathrm{d}x \\ &= \mathrm{d}x\;[z_0\,(a\;j + b) + (a\,r + b\,c)]\,/(j\,z_0 + r)\;\;. \end{aligned} \tag{6.70}$$

Next, multiply the resultant equation (6.70) by $\frac{(j\,z_0 + r)}{[z_0\,(a\;j + b) + (a\,r + b\,c)]}$.

$$\left[\frac{j\,z_0 + r}{z_0\,(a\;j + b) + (a\,r + b\,c)}\right]\mathrm{d}z_0 = \mathrm{d}x. \tag{6.71}$$

As a result, the left- and the right-hand sides are functions of only one variable each: that is z_0 and x. Thus, one can directly integrate both sides to determine one as a function of the other. Finally, because z_0 is equal to $(a\,x + b\,y)$, in this manner one has determined the desired solution $y \equiv y(x)$.

Equation (VII) : Solution

Comparing (VII) and its surrogate equation (6.67), one finds the relationships: $a \to 1\,;\; b \to 2\,;\; c \to 3\,;\; j \to 4\,;\; r \to 5$. Accordingly, (6.71) becomes

$$\left[\frac{4\,z_0 + 5}{6\,z_0 + 11}\right]\mathrm{d}z_0 = \mathrm{d}x\;. \tag{6.72}$$

Integrating both sides leads to

$$\frac{2\,z_0}{3} - \frac{7}{18}\log(6\,z_0 + 11) + \sigma_0 = x\;.$$

Setting $z_0 \;=\; (a\,x + b\,y) \to (x + 2\,y)$, one finds the requisite solution to (VII).

$$(VII):\;\; x = \frac{2(x + 2\,y)}{3} - \frac{7}{18}\log\,(6\,x + 12\,y + 11) + \sigma_0. \tag{6.73}$$

6.5.3 Problems Group V

Solve the following problems, labeled (1)–(7), that are in fact reducible to the form $q = \Phi\left(\frac{y}{x}\right)$. [Hint: Read (6.42)–(6.73).]

$$(1): \quad \frac{\mathrm{d}y}{\mathrm{d}x} = 1 + \left(\frac{y}{x+1}\right) + \left(\frac{y}{x+1}\right)^2 .$$

$$(2): \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \left[\frac{y+2}{x+3}\right] + 1 \ .$$

$$(3): \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \tan\left[\frac{y+1}{x+2}\right] + \left[\frac{y+1}{x+2}\right].$$

$$(4): \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \exp\left[-\left(\frac{y+2}{x+3}\right)\right] + \left(\frac{y+2}{x+3}\right) \ .$$

$$(5): \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{x-y+1}{x+2y+3}\right).$$

$$(6): \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{x-y+1}{x+3y+4}\right).$$

$$(7): \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \left[\frac{(x+y)+2}{3(x+y)+4}\right].$$

6.6 Exact Differential and Exact Differential Equation

6.6.1 Exact Differential

Consider variables X and Y that are both real and finite within an (X, Y) domain Δ, and a function $Z(X, Y)$ that has continuous partial derivatives $\left(\frac{\partial Z}{\partial X}\right)_Y$ and $\left(\frac{\partial Z}{\partial Y}\right)_X$. Then, the total derivative of Z, namely $\mathrm{d}Z(X, Y)$,

$$\mathrm{d}Z(X, Y) = U(X, Y)\,\mathrm{d}X + V(X, Y)\,\mathrm{d}Y \ , \tag{6.74}$$

is exact if $U(X, Y)$ and $V(X, Y)$ are functions of X and Y in the domain Δ and obey the so-called integrability requirement:

$$U(X, Y) = \left(\frac{\partial Z}{\partial X}\right)_Y \quad ; \quad V(X, Y) = \left(\frac{\partial Z}{\partial Y}\right)_X \ . \tag{6.75}$$

This requirement holds if the second mixed derivatives of Z are equal. That is, if the following is true.

$$\frac{\partial^2 Z}{\partial Y \partial X} = \frac{\partial^2 Z}{\partial X \partial Y}. \tag{6.76}$$

Thus, in two dimensions, the standard representation for an exact differential is

$$\mathrm{d}Z(X,Y) = \left(\frac{\partial Z}{\partial X}\right)_Y dX + \left(\frac{\partial Z}{\partial Y}\right)_X \mathrm{d}Y \tag{6.77}$$

with an important caveat that the following equality holds.

$$\left(\frac{\partial \left(\frac{\partial Z}{\partial X}\right)_Y}{\partial Y}\right)_X \equiv \frac{\partial^2 Z}{\partial Y \partial X} = \frac{\partial^2 Z}{\partial X \partial Y} \equiv \left(\frac{\partial \left(\frac{\partial Z}{\partial Y}\right)_X}{\partial X}\right)_Y \quad . \tag{6.78}$$

6.6.2 Exact Differential Equation

Given an exact differential $\mathrm{d}Z(X,Y)$, the relationship

$$\mathrm{d}Z(X,Y) = 0 \ , \tag{6.79}$$

or equivalently

$$\left(\frac{\partial Z}{\partial X}\right)_Y \mathrm{d}X + \left(\frac{\partial Z}{\partial Y}\right)_X \mathrm{d}Y = 0 \ , \tag{6.80}$$

is referred to as an exact differential equation.

6.6.3 Requirement

Consider differential equation (6.81).

$$\begin{aligned} \mathrm{d}Z(X,Y) &= U(X,Y)\,\mathrm{d}X + V(X,Y)\,\mathrm{d}Y \\ &= 0 \ . \end{aligned} \tag{6.81}$$

Its exactness requires (6.82) to be satisfied.

$$\left(\frac{\partial Z}{\partial X}\right)_Y = U(X,Y) \ ; \quad \left(\frac{\partial Z}{\partial Y}\right)_X = V(X,Y) \ ; \quad \left(\frac{\partial U}{\partial Y}\right)_X = \left(\frac{\partial V}{\partial X}\right)_Y \quad . \tag{6.82}$$

6.6.4 Solution

Consider differential equation (6.81)

$$\mathrm{d}Z(X,Y) = 0 \tag{6.83}$$

and assume it is exact. Solving it requires the evaluation of the function $Z(X, Y)$. To that purpose, proceed as follows: One has from (6.81) and (6.82):

$$U(X, Y) = \left(\frac{\partial Z}{\partial X}\right)_Y . \tag{6.84}$$

Integrating both sides with respect to X gives

$$\int U(X, Y)\, \mathrm{d}X = \int \left(\frac{\partial Z}{\partial X}\right)_Y \mathrm{d}X = Z(X, Y) + \mathcal{F}(Y) . \tag{6.85}$$

The function $\mathcal{F}(Y)$ is as yet undetermined. To eliminate it, proceed as follows. Equation (6.85) relates $\mathcal{F}(Y)$ to $\int U(X, Y)\, \mathrm{d}X$.

Symmetry considerations require the next task must be to relate $\mathcal{F}(Y)$ to $\int V(X, Y)\, \mathrm{d}Y$.

In order to do that, one needs to differentiate (6.85) with respect to Y while holding X constant.

$$\begin{aligned}\left(\frac{\partial[\int U(X, Y)\, \mathrm{d}X]}{\partial Y}\right)_X &= \left(\frac{\partial Z}{\partial Y}\right)_X + \frac{\mathrm{d}\mathcal{F}(Y)}{\mathrm{d}Y} \\ &= V(X, Y) + \frac{\mathrm{d}\mathcal{F}(Y)}{\mathrm{d}Y} .\end{aligned} \tag{6.86}$$

In the above, following (6.82), $\left(\frac{\partial Z}{\partial Y}\right)_X$ was replaced by $V(X, Y)$. This establishes the needed connection. And $\mathcal{F}(Y)$ can now be determined by integrating both sides of (6.86) with respect to Y.

$$\begin{aligned}\int \left(\frac{\partial[\int U(X, Y)\, \mathrm{d}X]}{\partial Y}\right)_X \mathrm{d}Y &= \int V(X, Y)\, \mathrm{d}Y + \int \frac{\mathrm{d}\mathcal{F}(Y)}{\mathrm{d}Y}\, \mathrm{d}Y + \sigma \\ &= \int V(X, Y)\, \mathrm{d}Y + \mathcal{F}(Y) + \sigma .\end{aligned} \tag{6.87}$$

Equations (6.85) and (6.87) are rewritten as

$$\begin{aligned}-\mathcal{F}(Y) &= -\int U(X, Y)\, \mathrm{d}X + Z(X, Y) \quad ; \\ -\mathcal{F}(Y) &= \int V(X, Y)\, \mathrm{d}Y - \int \left(\frac{\partial[\int U(X, Y)\, \mathrm{d}X]}{\partial Y}\right)_X \mathrm{d}Y + \sigma .\end{aligned}$$

Eliminate $-\mathcal{F}(Y)$. Thus, the solution $Z(X, Y)$ of the exact differential equation (6.81) is

$$Z(X,Y) = \int U(X,Y)\,\mathrm{d}X \; + \; \int V(X,Y)\,\mathrm{d}Y \\ - \int \left(\frac{\partial[\int U(X,Y)\,\mathrm{d}X]}{\partial Y} \right)_X \mathrm{d}Y + \sigma \; . \tag{6.88}$$

6.6.5 Examples Group VI

Given below are six differential equations that are similar to (6.81).

$$\begin{aligned}
(1): \; 0 \; &= \; \mathrm{d}Z = [1+2Y]\,\mathrm{d}X + [2+2X]\,\mathrm{d}Y \; . \\
(2): \; 0 \; &= \; \mathrm{d}Z = \left[3X^2+4XY\right]\mathrm{d}X + \left[2X^2+2Y\right]\mathrm{d}Y \; . \\
(3): \; 0 \; &= \; \mathrm{d}Z = \left[\exp(2Y)\right]\mathrm{d}X + \left[2X\exp(2Y) - Y\right]\mathrm{d}Y \; . \\
(4): \; 0 \; &= \; \mathrm{d}Z = \left[3YX^2 + \sin(Y)\right]\mathrm{d}X + \left[X\cos(Y) + X^3 + 1\right]\mathrm{d}Y \; . \\
(5): \; 0 \; &= \; \mathrm{d}Z = \left[\sin(Y) + X\,Y^2 + X^2\,Y + X + Y\right]\mathrm{d}X \\
&\qquad + \left[\frac{X^3}{3} + X\cos(Y) + X^2\,Y + X\right]\mathrm{d}Y \; . \\
(6): \; 0 \; &= \; \mathrm{d}Z = \left[X + 2X\cos(Y) + 3X^2Y\right]\mathrm{d}X \\
&\qquad + \left[X^3 - X^2\sin(Y) + Y\right]\mathrm{d}Y \; .
\end{aligned} \tag{6.89}$$

6.6.6 Solution

Choose $U(X,Y)$ and $V(X,Y)$ from one of the six equations (6.89). Check whether the relevant differential equation is exact by using (6.82). And if so, employ (6.88) to find its solution.

The results are given below ad seriatim.

$$\begin{aligned}
(1): \quad & U \; = \; 1+2Y \; ; \; V \; = \; 2+2X \; . \\
& \left(\frac{\partial U}{\partial Y}\right)_X \; = \; 2 \; = \left(\frac{\partial V}{\partial X}\right)_Y \; . \\
& \int U\,\mathrm{d}X \; = \; (X+2YX) \; , \quad \int V\,\mathrm{d}Y \; = \; (2Y+2XY) \; , \\
& -\int \left(\frac{\partial[\int U\,\mathrm{d}X]}{\partial Y} \right)_X \mathrm{d}Y \; = \; -\int 2X\,\mathrm{d}Y \; = \; -2X\,Y \; , \\
& \sigma_1 \; = \; (X+2YX) + (2Y+2XY) - (2XY) \\
& \quad \textit{Therefore} \; , \; Y \; = \; -\left(\frac{X-\sigma_1}{2+2X}\right) \qquad .
\end{aligned}$$

$$(2): \quad U = 3X^2 + 4XY \; ; \; V = 2X^2 + 2Y \; .$$
$$\left(\frac{\partial U}{\partial Y}\right)_X = 4X = \left(\frac{\partial V}{\partial X}\right)_Y \; .$$
$$\sigma_1 = (X^3 + 2YX^2) + (Y^2 + 2X^2Y) - (2X^2Y)$$
$$Therefore \; , \; Y^2 + 2X^2Y + X^3 - \sigma_1 = 0 \; .$$
$$Or \; equivalently \; , \; Y = -X^2 \pm \sqrt{X^4 - X^3 + \sigma_1} \; .$$

$$(3): \quad U = \exp(2Y) \; ; \; V = 2X\exp(2Y) - Y.$$
$$\left(\frac{\partial U}{\partial Y}\right)_X = 2\exp(2Y) = \left(\frac{\partial V}{\partial X}\right)_Y .$$
$$\int U \, dX = X\exp(2Y) \; , \quad \int V \, dY = X\exp(2Y) - Y^2/2,$$
$$-\int \left(\frac{\partial [\int U \, dX]}{\partial Y}\right)_X dY = -\int 2X \exp(2Y) \, dY = -X \exp(2Y),$$
$$\sigma_1 = X\exp(2Y) - Y^2/2$$
$$Therefore \; , X = (Y^2/2)\exp(-2Y) + (\sigma_1)\exp(-2Y) \; .$$

$$(4): \quad U = 3YX^2 + \sin(Y) \; ; \; V = X\cos(Y) + X^3 + 1 \; .$$
$$\left(\frac{\partial U}{\partial Y}\right)_X = 3X^2 + \cos(Y) = \left(\frac{\partial V}{\partial X}\right)_Y \; .$$
$$\sigma_1 = \left[X\sin(Y) + YX^3\right] + \left[X^3Y + X\sin(Y) + Y\right]$$
$$- \left[X\sin(Y) + YX^3\right]$$
$$Therefore \; , \; \sigma_1 = X^3Y + X\sin(Y) + Y \; .$$

$$(5): \quad U = \sin(Y) + X\,Y^2 + X^2\,Y + X + Y \; ;$$
$$V = \frac{X^3}{3} + X\cos(Y) + X^2\,Y + X \; .$$
$$\left(\frac{\partial U}{\partial Y}\right)_X = \cos(Y) + 2XY + X^2 + 1 = \left(\frac{\partial V}{\partial X}\right)_Y \; .$$
$$\sigma_1 = \left[X\sin(Y) + X^2Y^2/2 + X^3Y/3 + X^2/2 + XY\right]$$
$$+ X^3Y/3 + X\sin(Y) + X^2Y^2/2 + XY$$
$$- \left[X\sin(Y) + X^2Y^2/2 + X^3Y/3 + XY\right]$$
$$Therefore \; , \quad \sigma_1 = X^3Y/3 + X\sin(Y) + X^2Y^2/2 + XY + X^2/2 \; .$$

$$
\begin{aligned}
(6):\quad & U = 2X\cos(Y) + 3X^2Y + X\,;\; V = X^3 - X^2\sin(Y) + Y\,. \\
& \left(\frac{\partial U}{\partial Y}\right)_X = 3X^2 - 2X\sin(Y) = \left(\frac{\partial U}{\partial Y}\right)_X\,. \\
& \sigma_1 = \left[X^2\cos(Y) + X^3Y + X^2/2\right] + X^3Y + X^2\cos(Y) + Y^2/2 \\
& \quad - \left[X^2\cos(Y) + X^3Y\right]\,. \\
& Therefore\,,\ \sigma_1 = X^3Y + X^2\cos(Y) + X^2/2 + Y^2/2\,.
\end{aligned} \tag{6.90}
$$

Exercise

Due to symmetry, it is clear that the functions $U(X, Y)$ and $V(X, Y)$ may be interchanged as long as the same is done for the variables X and Y. Therefore, (6.88) can also be represented as

$$
\begin{aligned}
Z(X, Y) = & \int V(X, Y)\,\mathrm{d}Y + \int U(X, Y)\,\mathrm{d}X \\
& - \int \left(\frac{\partial[\int V(X, Y)\,\mathrm{d}Y]}{\partial X}\right)_Y \mathrm{d}X + \sigma\,.
\end{aligned} \tag{6.91}
$$

Show that this is true. Also, by using (6.91)—that is, instead of (6.88)—solve the six differential equations given as (6.89).

6.6.7 Problems Group VI

Solve the exact differential equations specified in problems (1) → (3). [Hint: Read (6.88) and (6.90).]

$$
\begin{aligned}
(1):\ 0 &= \mathrm{d}z = [1 + 2x\,y]\,\mathrm{d}x + \left[2 + x^2\right]\mathrm{d}y\,. \\
(2):\ 0 &= \mathrm{d}z = \left[1 + y + y^2\right]\mathrm{d}x + [2 + 2x\,y]\,\mathrm{d}y\,. \\
(3):\ 0 &= \mathrm{d}z = \left[\exp(y) + x\right]\mathrm{d}x + \left[x\exp(y) + y\right]\mathrm{d}y\,.
\end{aligned}
$$

6.7 Inexact Differential Equation Integrating Factor

A differential equation of the form

$$
u(x, y)\,\mathrm{d}x + v(x, y)\,\mathrm{d}y = 0\,, \tag{6.92}
$$

that is not exact, meaning

$$\left(\frac{\partial u}{\partial y}\right)_x \neq \left(\frac{\partial v}{\partial x}\right)_y \quad , \tag{6.93}$$

may be made exact if multiplied by an appropriate function $I(x, y)$. When that can be done, and the new differential equation

$$I(x, y)\,[u(x, y)\,\mathrm{d}x + v(x, y)\,\mathrm{d}y] = 0 \quad , \tag{6.94}$$

is exact, the multiplying function $I(x, y)$ is called an integrating factor.

Although, in principle, for every inexact differential equation there exists an appropriate integrating factor, in practice such integrating factors are hard to find. Moreover, a general rule for determining the relevant integrating factor is not available. However, if the integrating factor should turn out to be a function of only a single variable, progress can be made.

6.7.1 Integrating Factor Dependent only on x

Assume the integrating factor for the inexact differential equation (6.92) is dependent only on x. That is

$$I(x, y) \equiv X(x) \quad . \tag{6.95}$$

In other words, the following differential equation is assumed to be exact

$$X(x).[u\,\mathrm{d}x + v\,\mathrm{d}y] = 0 \quad . \tag{6.96}$$

Clearly, if (6.96) is indeed exact, it must satisfy the exactness rule

$$\left(\frac{\partial [X(x).u]}{\partial y}\right)_x = \left(\frac{\partial [X(x).v]}{\partial x}\right)_y \quad . \tag{6.97}$$

That is

$$X(x)\left(\frac{\partial u}{\partial y}\right)_x = X(x)\left(\frac{\partial v}{\partial x}\right)_y + v(x, y)\frac{\mathrm{d}X(x)}{\mathrm{d}x} \quad . \tag{6.98}$$

This readily leads to the relationship

$$\frac{\left(\frac{\partial u}{\partial y}\right)_x - \left(\frac{\partial v}{\partial x}\right)_y}{v(x, y)} = \frac{\frac{\mathrm{d}X(x)}{\mathrm{d}x}}{X(x)} \quad . \tag{6.99}$$

In other words, if the integrating factor X that makes the inexact differential equation (6.92) exact is to depend only on x, it must satisfy (6.99).

Multiplying both sides of (6.99) by $\mathrm{d}x$ and integrating gives

$$\int \frac{\left(\frac{\partial u}{\partial y}\right)_x - \left(\frac{\partial v}{\partial x}\right)_y}{v(x,\,y)} \mathrm{d}x = \log X(x) + \textit{constant} \,. \tag{6.100}$$

To conclude, if there is an integrating factor $X(x)$ that makes an inexact differential equation $u(x,\,y)\,\mathrm{d}x + v(x,\,y)\,\mathrm{d}y = 0$ exact, it must have the form

$$X(x) = \sigma_1 \exp\left[\int \frac{\left(\frac{\partial u(x,y)}{\partial y}\right)_x - \left(\frac{\partial v(x,y)}{\partial x}\right)_y}{v(x,\,y)} \mathrm{d}x\right] \,. \tag{6.101}$$

6.7.2 *Integrating Factor Dependent only on y*

Assume there is an integrating function, $Y(y)$, dependent only on y that makes the inexact differential equation (6.92) exact. That is, assume the following is an exact differential equation

$$Y(y)\,[u(x,\,y)\,\mathrm{d}x + v(x,\,y)\,\mathrm{d}y] = 0 \quad . \tag{6.102}$$

Comparing with (6.99) and (6.101), symmetry considerations suggest that if there is an integrating factor $Y(y)$ that makes an inexact differential equation $u\,\mathrm{d}x + v\,\mathrm{d}y = 0$ exact, it must have the form

$$\frac{\left(\frac{\partial v}{\partial x}\right)_y - \left(\frac{\partial u}{\partial y}\right)_x}{u(x,\,y)} = \frac{\frac{\mathrm{d}Y(y)}{\mathrm{d}y}}{Y(y)} \quad ;$$

$$Y(y) = \sigma_2 \exp\left[\int \frac{\left(\frac{\partial v(x,y)}{\partial x}\right)_y - \left(\frac{\partial u(x,y)}{\partial y}\right)_x}{u(x,\,y)} \mathrm{d}y\right] \,. \tag{6.103}$$

Exercise

Prove that (6.103), guessed on symmetry grounds, are correct.

6.7.3 *Examples Group VII*

Exact or Inexact?

Solve the following three differential equations.

$$\begin{aligned}(1):\ 0 &= \mathrm{d}z = [1+2y]\,\mathrm{d}x + [1+3x]\,\mathrm{d}y\ .\\ (2):\ 0 &= \mathrm{d}z = y^2\,\mathrm{d}x + 3xy\,\mathrm{d}y\ .\\ (3):\ 0 &= \mathrm{d}z = \left[x^2 - y\right]\mathrm{d}x + x\,\mathrm{d}y\ .\end{aligned} \tag{6.104}$$

Procedure: First choose $u(x, y)$ and $v(x, y)$ from one of the three equations (6.104). Then by using (6.82), check as to whether the relevant differential equation is exact or inexact. And if it is inexact, employ (6.99) or (6.103) to find whether an integrating factor that depends only upon a single variable is possible.

Solution of (6.104)-(1)

$$u = 1+2y\,;\ v = 1+3x\ .\ \left(\frac{\partial u}{\partial y}\right)_x = 2\ \textit{and}\ \left(\frac{\partial v}{\partial x}\right)_y = 3\ . \tag{6.105}$$

Because $2 \neq 3$, this equation is inexact. Therefore, try using (6.99) to check as to whether an integrating factor that depends only on the variable x is possible.

$$\frac{\left[\left(\frac{\partial u}{\partial y}\right)_x - \left(\frac{\partial v}{\partial x}\right)_y\right]}{v}.\mathrm{d}x = \left[\frac{-\mathrm{d}x}{1+3x}\right] = \left[\frac{\mathrm{d}X(x)}{X(x)}\right]\ . \tag{6.106}$$

Good news! There exists an integrating factor $X(x)$. One can find it by integrating the above equation.

$$\log[X(x)] = \log(1+3x)^{\frac{-1}{3}} + \textit{constant}. \tag{6.107}$$

Thus

$$X(x) = \frac{\sigma_0}{(1+3x)^{\frac{1}{3}}}\ . \tag{6.108}$$

Including the integrating factor, and ignoring the unnecessary multiplier σ_0, the differential equation (6.104)-(1) now is

$$(1+3x)^{-\frac{1}{3}}\left[(1+2y)\,\mathrm{d}x + (1+3x)\,\mathrm{d}y\right] = 0\ . \tag{6.109}$$

Now, determine whether this new differential equation is indeed exact. One has

$$\begin{aligned}u(x,y) &= (1+2y)(1+3x)^{-\frac{1}{3}}\ ;\ v(x,y) = (1+3x)^{\frac{2}{3}}.\ \textit{Therefore},\\ \left(\frac{\partial u}{\partial y}\right)_x &= 2(1+3x)^{-\frac{1}{3}}\ ;\ \left(\frac{\partial v}{\partial x}\right)_u = \frac{2}{3}\left[(1+3x)^{-\frac{1}{3}}\right]3\ .\end{aligned} \tag{6.110}$$

Good! The new equation is exact. Accordingly, one can find its solution via the procedure outlined in (6.88). That is

$$\begin{aligned}\sigma_1 - \sigma_2 &= \int u(x,y)\,\mathrm{d}x \;+\; \int v(x,y)\,\mathrm{d}y - \int \left(\frac{\partial[\int u(x,y)\,\mathrm{d}x]}{\partial y}\right)_x \mathrm{d}y \\ &= \int (1+3x)^{-\frac{1}{3}}(1+2y)\mathrm{d}x + \int (1+3x)^{\frac{2}{3}}\mathrm{d}y \\ &\quad - \int \left(\frac{\partial[\int u(x,y)\,\mathrm{d}x]}{\partial y}\right)_x \mathrm{d}y \\ &= \frac{(1+2y)(1+3x)^{\frac{2}{3}}}{\left[3\left(\frac{2}{3}\right)\right]} + (1+3x)^{\frac{2}{3}}y - (1+3x)^{\frac{2}{3}}y \\ &= \frac{(1+2y)(1+3x)^{\frac{2}{3}}}{2}. \end{aligned} \tag{6.111}$$

Therefore, the solution to (6.104)-(1) is

$$y(x) \;=\; \frac{-1}{2} + \frac{\sigma_1 - \sigma_2}{(1+3x)^{\frac{2}{3}}}\;. \tag{6.112}$$

Solution of (6.104)-(2)

$$u \;=\; y^2\,;\; v \;=\; 3xy \;;\quad \left(\frac{\partial u}{\partial y}\right)_x \;=\; 2y \;\neq\; \left(\frac{\partial v}{\partial x}\right)_y \;=\; 3y\,. \tag{6.113}$$

This equation is clearly NOT exact. However, to check as to whether an integrating factor that depends only on the variable y is possible, use (6.103).

$$\begin{aligned} \frac{\left[\left(\frac{\partial v}{\partial x}\right)_y - \left(\frac{\partial u}{\partial y}\right)_x\right]}{u}.\mathrm{d}y &= \frac{\left[\left(\frac{\mathrm{d}Y}{\mathrm{d}y}\right)_x\right]}{Y}.\mathrm{d}y \\ \left[\frac{3y-2y}{y^2}\right].\mathrm{d}y &= \frac{\mathrm{d}Y(y)}{Y(y)}\;. \end{aligned} \tag{6.114}$$

Good news! There exists an integrating factor $Y(y)$, and one can find it by integrating the above equation.

$$\log(y) = \log Y(y) \;+\; constant. \tag{6.115}$$

Thus

$$Y(y) = \sigma_0\, y\;. \tag{6.116}$$

With this integrating factor, the differential equation (6.104)-(2) becomes

$$y^3\,\mathrm{d}x + 3xy^2\,\mathrm{d}y \;=\; 0\;. \tag{6.117}$$

As before one checks as to whether this new differential equation is indeed exact.

$$u(x,y) = y^3 \; ; \; v(x,y) = 3y^2x \; .$$
$$And, \quad \left(\frac{\partial u}{\partial y}\right)_x = 3y^2 \quad ; \quad \left(\frac{\partial v}{\partial x}\right)_u = 3y^2 \; . \tag{6.118}$$

Yes, the new equation is exact. Accordingly, one can find its solution via the procedure outlined in (6.91). That is

$$\begin{aligned}\sigma_1 - \sigma_2 &= \int v(x,y)\,dy + \int u(x,y)\,dx - \int \left(\frac{\partial[\int v(x,y)\,dy]}{\partial x}\right)_y dx \\ &= \int (3y^2x)\,dy + \int (y^3)\,dx - \int \left(\frac{\partial[\int v(x,y)\,dy]}{\partial x}\right)_y dx \\ &= y^3x + y^3x - y^3x = y^3x \; .\end{aligned} \tag{6.119}$$

Accordingly, the solution to (6.104)-(2) is

$$y(x) = \left[\frac{\sigma_1 - \sigma_2}{x}\right]^{\frac{1}{3}} \; . \tag{6.120}$$

Solution of (6.104)-(3)

$$u = x^2 - y \; ; \; v = x \; ; \quad \left(\frac{\partial u}{\partial y}\right)_x = -1 \neq \left(\frac{\partial v}{\partial x}\right)_y = 1 \; . \tag{6.121}$$

This equation is NOT exact. Therefore, try (6.99) to check as to whether an integrating factor that depends only on one variable is possible. To that end, x would appear to be the more likely candidate.

$$\frac{\left[\left(\frac{\partial u}{\partial y}\right)_x - \left(\frac{\partial v}{\partial x}\right)_y\right]}{v}.dx = \left[\frac{-2\,dx}{x}\right] = \left[\frac{dX}{X}\right] \; . \tag{6.122}$$

Thus, there appears to be an integrating factor X that depends only on x. And it obeys the relationship

$$\log(X) = -2\log x + constant \tag{6.123}$$

leading to

$$X(x) = \frac{\sigma_0}{x^2} \; . \tag{6.124}$$

With the integrating factor included, the differential equation (6.104)-(3) reads

$$\begin{aligned} &\frac{1}{x^2}\left[\left(x^2-y\right)\,\mathrm{d}x + x\,\mathrm{d}y\right] \;=\; 0 \;; \\ &u(x,y) \;=\; \frac{1}{x^2}\left(x^2-y\right) \;\;;\;\; v(x,y) \;=\; \frac{1}{x^2}\,x \;=\; \frac{1}{x} \;. \end{aligned} \tag{6.125}$$

For exactness, it must satisfy the equality

$$\left(\frac{\partial u}{\partial y}\right)_x \;=\; -\frac{1}{x^2} = \left(\frac{\partial v}{\partial x}\right)_u \tag{6.126}$$

which it does. Accordingly, one can find its solution via the procedure outlined in (6.88). That is

$$\begin{aligned} \sigma_1 - \sigma_2 &= \int u(x,y)\,\mathrm{d}x \;+\; \int v(x,y)\,\mathrm{d}y - \int \left(\frac{\partial[\int u(x,y)\,\mathrm{d}x]}{\partial y}\right)_x \mathrm{d}y \\ &= \left(x+\frac{y}{x}\right) + \frac{y}{x} - \frac{y}{x} \;=\; \left(x+\frac{y}{x}\right) \;. \end{aligned} \tag{6.127}$$

Therefore, the solution to (6.104)-(3) is

$$y(x) \;=\; (\sigma_1-\sigma_2)x \;-\; x^2. \tag{6.128}$$

6.7.4 *Problems Group VII*

Solve the following three differential equations.
[Hint: Compare (6.81)–(6.128).]

$$\begin{aligned} &(1):\; 0 \;=\; \mathrm{d}z = (2+3y)\;\mathrm{d}x + (3+2x)\;\mathrm{d}y \;. \\ &(2):\; 0 \;=\; \mathrm{d}z = y^3\,\mathrm{d}x + 2xy^2\,\mathrm{d}y \;. \\ &(3):\; 0 \;=\; \mathrm{d}z = \left(x^2+y\right)\,\mathrm{d}x + \left(\frac{1}{3x}\right)\mathrm{d}y. \end{aligned} \tag{6.129}$$

6.8 Riccati Equation

A nonlinear differential equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 + f_1(x)\,y + f_2(x) \;=\; 0 \tag{6.130}$$

is sometime called a Riccati equation.

6.8.1 Treatment

Set

$$\frac{1}{z}\,\frac{\mathrm{d}z}{\mathrm{d}x} \;=\; y \;. \tag{6.131}$$

As a result,

$$\frac{\mathrm{d}z}{\mathrm{d}x} \;=\; z\,y \;. \tag{6.132}$$

Its differentiation gives

$$\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} \;=\; z\,\frac{\mathrm{d}y}{\mathrm{d}x} + y\,\frac{\mathrm{d}z}{\mathrm{d}x} = z\,\frac{\mathrm{d}y}{\mathrm{d}x} + y^2\,z \;. \tag{6.133}$$

Now, multiply (6.130) by z

$$z\left[\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 + f_1(x)\,y + f_2(x)\right] \;=\; 0 \tag{6.134}$$

and replace $z\,\frac{\mathrm{d}y}{\mathrm{d}x}$ according to (6.133). This gives

$$\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} - y^2\,z \;+ z\,y^2 + f_1(x)\,z\,y + f_2(x)\,z \;=\; 0 \;. \tag{6.135}$$

Remove $-y^2\,z \;+ z\,y^2$ and replace $z\,y$ by $\frac{\mathrm{d}z}{\mathrm{d}x}$. As a result, the Riccati equation (6.130) is transformed into

$$\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} + f_1(x)\,\frac{\mathrm{d}z}{\mathrm{d}x} + f_2(x)z \;=\; 0 \;. \tag{6.136}$$

Notice that this differential equation, unlike the original Riccati equation, is linear. But the removal of nonlinearity has resulted in increasing the order. And we now have a second-order differential equation. But, because the new equation is linear, there is a greater possibility of making progress. In particular, if $f_1(x)$ and $f_2(x)$ should both turn out to be constants, (6.136) can be solved much like equations of the type (3.1), (3.5), etc. And once z has been determined, y can be calculated by a simple differentiation as is implicit in the form of (6.131).

6.8.2 Examples Group VIII

First convert the Riccati equations given below into second-order linear equations and then solve them.

$$\begin{aligned}
(1): &\quad \frac{\mathrm{d}y}{\mathrm{d}x} + y^2 + x\,y = 0.\\
(2): &\quad \frac{\mathrm{d}y}{\mathrm{d}x} + y^2 + x^2\,y = 0.\\
(3): &\quad \frac{\mathrm{d}y}{\mathrm{d}x} + y^2 + y + 1 = 0.\\
(4): &\quad \frac{\mathrm{d}y}{\mathrm{d}x} + y^2 + 2\,y + 2 = 0.
\end{aligned} \tag{6.137}$$

Converting the Riccati equations into second-order linODE's is straightforward. All one needs to do is use (6.130) and (6.136). Accordingly, (6.137) get transformed as follows:

$$\begin{aligned}
(1) \;\rightarrow\; (1'): &\quad \tfrac{\mathrm{d}^2z}{\mathrm{d}x^2} + x\,\tfrac{\mathrm{d}z}{\mathrm{d}x} = 0\\
(2) \;\rightarrow\; (2'): &\quad \tfrac{\mathrm{d}^2z}{\mathrm{d}x^2} + x^2\,\tfrac{\mathrm{d}z}{\mathrm{d}x} = 0\\
(3) \;\rightarrow\; (3'): &\quad \tfrac{\mathrm{d}^2z}{\mathrm{d}x^2} + \tfrac{\mathrm{d}z}{\mathrm{d}x} + z = 0\\
(4) \;\rightarrow\; (4'): &\quad \tfrac{\mathrm{d}^2z}{\mathrm{d}x^2} + 2\,\tfrac{\mathrm{d}z}{\mathrm{d}x} + 2\,z = 0
\end{aligned} \tag{6.138}$$

6.8.3 Solution

Equations (6.137)-(1) **and** (6.138)-(1')

Introduce the notation $p = \frac{\mathrm{d}z}{\mathrm{d}x}$ and write (6.250)-(1') as

$$\frac{\mathrm{d}p}{\mathrm{d}x} + x\,p = 0\ . \tag{6.139}$$

Equation (6.139) is readily solved. (See, for instance, (4.2), (4.10), and (4.15) that describe how such first-order linODEs are solved.) That is

$$p = \sigma_0 \exp\left(-\frac{x^2}{2}\right)\ . \tag{6.140}$$

Because $p = \frac{\mathrm{d}z}{\mathrm{d}x}$, its integration leads to z.

$$\int \frac{\mathrm{d}z}{\mathrm{d}x}\,\mathrm{d}x = z = \int p\,\mathrm{d}x = \sigma_0 \int \exp\left(-\frac{x^2}{2}\right)\mathrm{d}x + \sigma_1\ . \tag{6.141}$$

And once z is known, y can be determined by using (6.131). That is

$$y = \frac{1}{z}\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{p}{z} = \frac{\sigma_0 \exp\left(-\frac{x^2}{2}\right)}{\sigma_0 \int \exp\left(-\frac{x^2}{2}\right)\mathrm{d}x + \sigma_1}$$
$$= \frac{\exp\left(-\frac{x^2}{2}\right)}{\int \exp\left(-\frac{x^2}{2}\right)\mathrm{d}x + \sigma_2} \ . \tag{6.142}$$

Equations (6.137)-(2) **and** (6.138)-(2')

Follow the same procedure as in (6.139)–(6.142) and write (6.138)-(2') as

$$\frac{\mathrm{d}p}{\mathrm{d}x} + x^2\, p = 0. \tag{6.143}$$

Again with the help of (4.2), (4.10), and (4.15) one gets

$$p = \sigma_0 \exp\left(-\frac{x^3}{3}\right). \tag{6.144}$$

Because $p = \frac{\mathrm{d}z}{\mathrm{d}x}$ its integration leads to z.

$$\int \mathrm{d}z = z = \sigma_0 \int \exp\left(-\frac{x^3}{3}\right)\mathrm{d}x + \sigma_1. \tag{6.145}$$

And once z is known, y can be determined by using (6.131). Much like (6.142) one can write y as

$$y = \frac{\exp\left(-\frac{x^3}{3}\right)}{\int \exp\left(-\frac{x^3}{3}\right)\mathrm{d}x + \sigma_2}. \tag{6.146}$$

Equations (6.137)-(3) **and** (6.138)-(3')

Notice that (6.138)-(3') is homogeneous linear ordinary differential equation with constant coefficients.

And according to (3.13), the complimentary solution of a second-order homogeneous linear ordinary differential equation with constant coefficients can be written as

$$z \equiv S_{comp}(x) = \sigma_1 \exp(k_1 x) + \sigma_2 \exp(k_2 x) \ . \tag{6.147}$$

Here, $k = k_1$ and $k = k_2$ are solutions of the $Characteristic\ Equation(E_{ch})$ for (6.138)-(3'). The procedure for finding E_{ch} is to replace z by 1, $\frac{dz}{dx}$ by k, and $\frac{d^2z}{dz^2}$ by k^2. Thus, the $S_{comp}(x)$ for (6.138)-(3') is found by solving the following E_{ch}.

$$k^2 + k + 1 = 0 \tag{6.148}$$

which leads to

$$\begin{aligned} k &= k_1 = r + im = \frac{-1 + i\sqrt{3}}{2}, \\ k &= k_2 = r - im = \frac{-1 - i\sqrt{3}}{2}. \end{aligned} \tag{6.149}$$

Therefore, using (6.147)–(6.149), the result for z, according to (3.21) and (3.22), can be written as

$$\begin{aligned} z &= \sigma_1 \exp(r - i\,m)\,x + \sigma_2 \exp(r + i\,m)x \\ &= \exp(r\,x)\left[\sigma_1 \exp(-i\,m\,x) + \sigma_2 \exp(i\,m\,x)\right] \\ &= \exp(r\,x)\left[\sigma_3 \sin(m\,x) + \sigma_4 \cos(m\,x)\right] \\ &= \exp\left(-\frac{x}{2}\right)\left[\sigma_3 \sin\left(\frac{\sqrt{3}x}{2}\right) + \sigma_4 \cos\left(\frac{\sqrt{3}x}{2}\right)\right] . \end{aligned} \tag{6.150}$$

Or, alternatively, as

$$z = \sigma_0 \exp\left(-\frac{x}{2}\right)\cos\left(\sigma_5 - \frac{\sqrt{3}}{2}x\right) . \tag{6.151}$$

Note, one went from (6.150)–(6.151) by introducing two new arbitrary constants σ_0 and σ_5 such that $\sigma_3 = \sigma_0 \sin(\sigma_5)$ and $\sigma_4 = \sigma_0 \cos(\sigma_5)$. This is perfectly alright as long as $\sigma_0^2 = \sigma_3^2 + \sigma_4^2$. And because all sigmas are arbitrary constants, this equality is trivially achieved.

Given z—see, (6.151)—the solution y is readily found.

$$y = \frac{1}{z}\frac{dz}{dx} = -\frac{1}{2} + \frac{\sqrt{3}}{2}\tan\left(\sigma_5 - \frac{\sqrt{3}}{2}x\right) . \tag{6.152}$$

Equations (6.137)-(4) **and** (6.138)-(4')

Again follow a similar procedure to that used for (6.147)–(6.152) and write the E_{ch} for (6.138)-(4') as

$$k^2 + 2k + 2 = 0 \tag{6.153}$$

which leads to

$$\begin{aligned} k &= k_1 = r + im = -1 + i \ , \\ k &= k_2 = r - im = -1 - i \ . \end{aligned} \tag{6.154}$$

Therefore,

$$\begin{aligned} z &= \exp(-x)\,[\sigma_3 \sin x + \sigma_4 \cos x] \\ &\equiv \sigma_0 \exp(-x) \cos(\sigma_5 - x) \ . \end{aligned} \tag{6.155}$$

And the solution y is

$$y = \frac{1}{z}\frac{\mathrm{d}z}{\mathrm{d}x} = -1 + \tan(\sigma_5 - x) \ . \tag{6.156}$$

6.8.4 *Problems Group VIII*

Solve the following two problems that involve Riccati equations.
[Hint: Read (6.137)–(6.156).]

$$(1): \qquad \frac{\mathrm{d}y}{\mathrm{d}x} + y^2 + 2\,y + 1 = 0.$$

$$(2): \qquad \frac{\mathrm{d}y}{\mathrm{d}x} + y^2 + y + 2 = 0.$$

6.9 Euler Equation

Among Euler's multifarious contributions to mathematics, in particular to the theory of differential equations, is his work related to linear ordinary differential equations with constant coefficient. He showed how linear ordinary differential equations with variable coefficients of the form, $c_n\, x^n$, given below

$$\sum_{n=0}^{\nu} c_n\, x^n\, D^n u = B(x) \ , \tag{6.157}$$

could be transformed into linear ordinary differential equations with constant coefficients. The latter are much easier to solve as was demonstrated in detail in Chaps. 3 and 4.

The Euler procedure consists in arranging an appropriate change of the independent variable that transforms the operator D into a new operator Δ. One proceeds as follows. Set

$$x \;=\; \exp(t)$$

and differentiate with respect to x.

$$1 = \exp(t) \,.\, \frac{\mathrm{d}t}{\mathrm{d}x} \quad . \tag{6.158}$$

Consider an arbitrary function $U(x,t)$ that is differentiable with respect to x and t. Multiply both sides of (6.158) from the right by $\frac{\mathrm{d}\,U(x,t)}{\mathrm{d}\,t}$.

$$\begin{aligned} 1 \cdot \frac{\mathrm{d}\,U(x,t)}{\mathrm{d}\,t} &= \exp(t) \,.\, \frac{\mathrm{d}t}{\mathrm{d}x} . \frac{\mathrm{d}U(x,t)}{\mathrm{d}\,t} \\ &= \exp(t) \cdot \frac{\mathrm{d}\,U(x,t)}{\mathrm{d}\,x} \\ = \frac{\mathrm{d}\,U(x,t)}{\mathrm{d}\,t} &= x \;\frac{\mathrm{d}\,U(x,t)}{\mathrm{d}\,x}. \end{aligned} \tag{6.159}$$

Because $U(x,t)$ is arbitrary, (6.159) implies $x \;\frac{\mathrm{d}}{\mathrm{d}\,x} \;=\; \frac{\mathrm{d}}{\mathrm{d}t}$: meaning

$$x \;D = \Delta. \tag{6.160}$$

Next, let us look at $x\,D\,(x\,D)$.

$$\begin{aligned} x\,D\,(x\,D) &= x\;(D\,x)\,D + x^2\,D^2 \\ = \quad \Delta\,(\Delta) &= x\,D + x^2\,D^2 = \Delta + x^2\,D^2 \quad . \end{aligned} \tag{6.161}$$

Therefore

$$x^2\,D^2 \equiv \;\Delta\,(\Delta - 1) \quad . \tag{6.162}$$

Similarly

$$x\,D\left[x^2\,D^2\right] = \Delta\,[\Delta\,(\Delta - 1)] \quad . \tag{6.163}$$

One also has

$$\begin{aligned} x\,D\left[x^2\,D^2\right] \;&=\; x\left(2\,x\,D^2\right) + x^3\,D^3 = 2\,x^2\,D^2 + x^3\,D^3 \\ = 2\,\Delta\,(\Delta - 1) + x^3\,D^3. \end{aligned} \tag{6.164}$$

The left-hand sides of (6.163) and (6.164) are equal. The equality of their right-hand sides yields

$$2\,\Delta\,(\Delta-1)+x^3\,D^3=\Delta\,[\Delta\,(\Delta-1)]\quad, \tag{6.165}$$

or

$$x^3\,D^3\equiv\Delta(\Delta-1)(\Delta-2)\quad. \tag{6.166}$$

In order to proceed to the next order, multiply $x^3\,D^3$ from the left by $x\,D$ and use (6.166) to appropriately replace one of the expressions equal to $3\,x^3\,D^3$.

$$\begin{aligned} x\,D\left[x^3\,D^3\right] &= x\left(3\,x^2\,D^3\right)+x^4\,D^4=3\,x^3\,D^3+x^4\,D^4 \\ &= 3\,\Delta\,(\Delta-1)\,(\Delta-2)+x^4\,D^4. \end{aligned} \tag{6.167}$$

Multiply the left-hand side of (6.166) by $x\,D$ and the right-hand side by the same amount, that is Δ. One gets

$$x\,D\left[x^3\,D^3\right]=\Delta\,[\Delta(\Delta-1)(\Delta-2)]\quad. \tag{6.168}$$

Now that the left-hand sides of (6.167) and (6.168) are the same, one can claim equality of their right-hand sides. That is

$$3\,\Delta\,(\Delta-1)\,(\Delta-2)+x^4\,D^4\;=\;\Delta\,[\Delta(\Delta-1)(\Delta-2)]\;,$$

which leads to the result

$$x^4\,D^4\equiv\Delta(\Delta-1)(\Delta-2)(\Delta-3)\quad. \tag{6.169}$$

Exercise

Clearly, (6.160), (6.162), (6.166), and (6.169) show a pattern. Therefore, by induction, one assumes

$$x^n\,D^n=\Delta(\Delta-1)...(\Delta-n+1)\quad. \tag{6.170}$$

Assuming (6.170) is valid when $n=\ell-1$, show that it is also valid when $n=\ell$.

6.9.1 Examples Group IX

Transform each of the four differential equations $IX-(A)\rightarrow IX-(D)$ given below by using the transformation $x=\exp(t)$, and renaming the function $u(x)$ as $U(t)$. Finally, represent the result in terms of x. Then, find $u(x)$.

$$IX-(A): \quad \left[x^2\,D^2+x\,D+1\right]u(x)=a\,x^2+b\ . \tag{6.171}$$

Solution IX-(A)

Use (6.158), (6.160), and (6.162) and transform (6.171) into

$$[\Delta(\Delta-1)+\Delta+1]\,U(t)\ =\ \left[\Delta^2+1\right]U(t)=a\exp(2\,t)+b\ . \tag{6.172}$$

Then, solve for $U(t)$ by following the familiar routine.

$$\begin{aligned}
E_{ch}\ &=\ k^2+1=0 \qquad ;\ k_{1,2}\ =\ \pm i\ ;\\
S_{comp}(t)\ &=\ \sigma_0\exp(-i\,t)\ +\ \sigma_i\exp(-i\,t)=\sigma_2\sin(t)+\sigma_3\cos(t)\ ;\\
I_{pi}(t)\ &=\ \frac{1}{\left[\Delta^2+1\right]}\{a\exp(2\,t)+b\}=a\exp(2\,t)\frac{1}{2^2+1}+\frac{b}{1}\ ;\\
U(t)\ &=\ S_{comp}(t)+I_{pi}(t)=\sigma_2\sin(t)\ +\sigma_3\cos(t)+\frac{a}{5}\exp(2\,t)+b\ .
\end{aligned}$$

Finally transform $U(t)$ back into $u(x)$. In this fashion, one gets:

$$U(t)\ \rightarrow\ u(x)\ =\ \sigma_2\sin[\log(x)]+\sigma_3\cos[\log(x)]+\frac{a}{5}x^2+b. \tag{6.173}$$

Solution IX-(B)

$$IX-(B): \quad \left[x^2\,D^2-2\,x\,D+4\right]u\ =\ a\log(x)\ . \tag{6.174}$$

As above use (6.158), (6.160), and (6.162) and transform (6.174) into the following differential equation.

$$\left[\Delta^2-3\Delta+4\right]U(t)=a\,t\ . \tag{6.175}$$

Its solution is found in the usual manner.

$$\begin{aligned}
E_{ch}\ &=\ k^2-3\,k+4=0 \qquad ;\ k_{1,2}\ =\frac{3}{2}\pm i\,\frac{\sqrt{7}}{2};\\
S_{comp}(t)&=\sigma_1\exp\left(\frac{3t}{2}\right)\sin\left(\frac{\sqrt{7}}{2}t\right)+\sigma_2\exp\left(\frac{3t}{2}\right)\cos\left(\frac{\sqrt{7}}{2}t\right);\\
I_{pi}(t)\ &=\ \frac{1}{\left[\Delta^2-3\Delta+4\right]}(a\,t)=\left(\frac{1}{4}+\frac{3}{16}\Delta\right)(a\,t)=\frac{a}{4}t+\frac{3a}{16};\\
U(t)\ &=\ S_{comp}(t)+I_{pi}(t)\ ;\\
U(t)\ \rightarrow\ u(x)\ &=\ \sigma_1\sin\left[\frac{\sqrt{7}}{2}\log(x)\right]x^{\frac{3}{2}}+\sigma_2\cos\left[\frac{\sqrt{7}}{2}\log(x)\right]x^{\frac{3}{2}}
\end{aligned}$$

$$+\frac{a}{4}\log(x)+\frac{3a}{16}. \tag{6.176}$$

Solution IX-(C)

$$IX-(C): \quad \left[x^2\,D^2+2x\,D-4\right]u \;=\; a\,x^2\,\log(x)+b\;. \tag{6.177}$$

Again use (6.158), (6.160), and (6.162) and transform (6.177) into

$$\left[\Delta^2+\Delta-4\right]U(t)=a\,\exp(2\,t)\,t+b\;, \tag{6.178}$$

Its solution is as follows:

$$\begin{aligned}
E_{ch} \;&=\; k^2+k-4=0 \qquad ;\; k_{1,2}\;=-\frac{1}{2}\pm\frac{\sqrt{17}}{2}\;;\\
S_{comp}(t)&=\sigma_1\exp\left(-\frac{t}{2}+\frac{\sqrt{17}}{2}t\right)+\sigma_2\exp\left(-\frac{t}{2}-\frac{\sqrt{17}}{2}t\right);\\
I_{pi}(t)\;&=\frac{1}{\left[\Delta^2+\Delta-4\right]}\{a\,\exp(2\,t)\,t\;+b\}=a\exp(2t)\frac{1}{\{(\Delta+2)^2+(\Delta+2)-4\}}t\\
+\left(\frac{b}{-4}\right)&=a\exp(2t)\left[\frac{1}{2+5\Delta+\Delta^2}\right]t-\frac{b}{4}=a\exp(2t)\left[\frac{1}{2}-\frac{5}{4}\Delta\right]t-\frac{b}{4};\\
I_{pi}(t)\;&=a\exp(2t)\left[\frac{1}{2}t-\frac{5}{4}\right]-\frac{b}{4};\\
U(t)\;&=S_{comp}(t)+I_{pi}(t);\\
U(t)\;\rightarrow\;u(x)\;&=\sigma_1\,x^{-\frac{1}{2}+\frac{\sqrt{17}}{2}}+\sigma_2\,x^{-\frac{1}{2}-\frac{\sqrt{17}}{2}}\\
&\quad+a\,x^2\left[\frac{1}{2}\log(x)-\frac{5}{4}\right].
\end{aligned} \tag{6.179}$$

Solution IX-(D)

$$IX-(D): \quad \left[x^2\,D^2-4x\,D+2\right]u \;=\; 2\,x\,\cos\left[\log(x)\right]. \tag{6.180}$$

And finally, as before, use (6.158), (6.160), and (6.162) and transform (6.180) into

$$\left[\Delta^2-5\Delta+2\right]U(t)=2\,\exp(t)\cos(t)\;. \tag{6.181}$$

This leads to

$$E_{ch} = k^2 - 5k + 2 \; ; \; k_{1,2} = \frac{5}{2} \pm \frac{\sqrt{17}}{2};$$

$$S_{comp}(t) = \sigma_1 \exp\left(\frac{5}{2}t + \frac{\sqrt{17}}{2}t\right) + \sigma_2 \exp\left(\frac{5}{2}t - \frac{\sqrt{17}}{2}t\right) \; ;$$

$$I_{pi}(t) = \frac{1}{\left[\Delta^2 - 5\Delta + 2\right]}\{2a \exp(t)\cos(t)\} = \frac{1}{\left[\Delta^2 - 5\Delta + 2\right]}\exp[t(1+i)] + \frac{1}{\left[\Delta^2 - 5\Delta + 2\right]}\exp[t(1-i)];$$

$$= \exp[t(1+i)]\left[\frac{1}{-3-3i}\right] + \exp[t(1-i)]\left[\frac{1}{-3+3i}\right]$$

$$= -\frac{\exp(t)}{6}\left[(1-i)\exp(i\,t) + (1+i)\exp(-i\,t)\right] = I_{pi}(t) = -\frac{1}{3}\exp(t)\left[\sin(t) + \cos(t)\right].$$

$$\begin{aligned} U(t) &= S_{comp}(t) + I_{pi}(t) \\ U(t) &\rightarrow u(x) = \sigma_1 x^{\frac{5}{2}+\frac{\sqrt{17}}{2}} + \sigma_2 x^{\frac{5}{2}-\frac{\sqrt{17}}{2}} \\ &-\frac{x}{3}\left[\sin\{\log(x)\} + \cos\{\log(x)\}\right]. \end{aligned} \tag{6.182}$$

6.9.2 Euler Equation: An Extension

A simple extension of Euler equation is given below in (6.183).

$$\sum_{n=0}^{\nu} c_n \, (a\,x + b)^n \, D^n u = B(x), \tag{6.183}$$

Any such linear ordinary differential equation with variable coefficients of the form $c_n \, (a\,x + b)^n$ can readily be transformed into a linear ordinary differential equation with constant coefficients—meaning into equations of the type that we have been studying previously. The relevant transformation is carried out by a change of the variable: from $(a\,x + b)$ to $\exp(a\,t)$ where both a and b are constants. That is

$$(a\,x + b) = \exp(a\,t). \tag{6.184}$$

Differentiate (6.184) with respect to x, and divide both sides by the constant a.

$$\begin{aligned} 1 &= \exp(a\,t) \cdot \frac{\mathrm{d}t}{\mathrm{d}x} \\ &= (a\,x + b) \cdot \frac{\mathrm{d}t}{\mathrm{d}x}. \end{aligned} \tag{6.185}$$

Consider an arbitrary function $U(x,t)$ that is differentiable with respect to x and t. Multiply both sides of (6.185) from the right by $\frac{\mathrm{d}U(x,t)}{\mathrm{d}t}$.

$$\begin{aligned}\frac{\mathrm{d}U(x,t)}{\mathrm{d}t} &= (a\,x+b)\,.\,\frac{\mathrm{d}t}{\mathrm{d}x}\,\frac{\mathrm{d}U(x,t)}{\mathrm{d}t}\\ &= (a\,x+b)\,.\,\frac{\mathrm{d}U(x,t)}{\mathrm{d}x}.\end{aligned} \tag{6.186}$$

Because $U(x,t)$ is arbitrary, this implies $\frac{\mathrm{d}}{\mathrm{d}t} = (a\,x+b)\,\frac{\mathrm{d}}{\mathrm{d}x}$: meaning

$$\Delta = (a\,x+b)\,D. \tag{6.187}$$

Next, let us look at $(a\,x+b)\,D\,\{(a\,x+b)\,D\}$ or in other words at $\Delta\,\{\Delta\}$.

$$\begin{aligned}(a\,x+b)\,D\,\{(a\,x+b)\,D\} &= (a\,x+b)[a\,D+(a\,x+b)\,D^2]\\ = \Delta\,\{\Delta\} \qquad &= \quad a\,\Delta+(a\,x+b)^2D^2 \quad .\end{aligned} \tag{6.188}$$

Transferring $a\,\Delta$ across to the left side gives

$$(a\,x+b)^2D^2 = \Delta\,(\Delta-a) \quad . \tag{6.189}$$

Similarly

$$\begin{aligned}(a\,x+b)\,D\left[(a\,x+b)^2\,D^2\right] &= (a\,x+b)\,\left(2a^2x+2ab\right)\,D^2+(a\,x+b)[(a\,x+b)^2\,D^3]\\ &= 2a(ax+b)^2D^2+(ax+b)^3D^3\\ &= 2\,a\,\Delta\,(\Delta-a)+(a\,x+b)^3\,D^3 \quad .\end{aligned} \tag{6.190}$$

By using (6.189) and the equivalence $(a\,x+b)\,D=\Delta$ on the left-hand side of (6.190), one gets

$$\Delta\,[\Delta\,(\Delta-a)] = 2\,a\,\Delta\,(\Delta-a)+(a\,x+b)^3\,D^3 \quad , \tag{6.191}$$

which leads to

$$(a\,x+b)^3\,D^3 = \Delta(\Delta-a)(\Delta-2\,a) \quad . \tag{6.192}$$

Again, starting with $(a\,x+b)\,D\left[(a\,x+b)^3\,D^3\right]$, one can readily show that

$$(a\,x+b)^4\,D^4 = \Delta(\Delta-a)(\Delta-2\,a)(\Delta-3\,a) \quad . \tag{6.193}$$

Also, in view of (6.187), (6.189), (6.192), (6.193), or equivalently (6.194) given below, one can transform a general equation of the type (6.183) into one involving Δ's and functions of the variable t. One must not forget, however, to use $(ax+b)=$

$\exp(at)$ for transforming out of the variable $B(x)$ into an appropriate function of the variable t. This will become clear in examples (XI) given below.

Exercise

Clearly, (6.187), (6.189), (6.192), and (6.193) show a pattern. Therefore, by induction

$$(a\,x+b)^n\,D^n = \Delta\{\Delta - a\}...\{\Delta - (n-1)a\}\ . \tag{6.194}$$

Assuming (6.194) is valid when $n = \ell - 1$, show that it is also valid when $n = \ell$.

6.9.3 Examples Group X

Extended Euler Equation

Transform each of the following three differential equations by setting the variable $(a\,x+b) = \exp(a\,t)$ and then find their solution $U(t)$. Also write the corresponding solution $u(x)$.

$$X(A):\ \left[(2x+3)^2\,D^2 + 2(2x+3)\,D - 4\right]u(x) = 5\log[2x+3]\,. \tag{6.195}$$

Solution of Example X(A)

Set $(2x+3) = \exp(2\,t)$ and $(2x+3)D = \Delta$, use (6.187) and (6.189), and transform (6.195) into the following:

$$\begin{aligned}[\Delta(\Delta-2) + 2\Delta - 4]\,U(t) &= 5\log\left[\exp(2t)\right]\\ = \left[\Delta^2 - 4\right]U(t) \qquad &= 10\,t\quad .\end{aligned} \tag{6.196}$$

Now proceed with the usual routine.

$$\begin{aligned} E_{ch} &= k^2 - 4 = 0 \quad ;\ k_{1,2} = \pm 2\ ;\\ S_{comp}(t) &= \sigma_0\exp(2\,t) + \sigma_1\exp(-2\,t)\ ;\\ I_{pi}(t) &= \frac{1}{\left[\Delta^2-4\right]}\{10\,t\} = -\frac{1}{4}\left[1+\frac{\Delta^2}{4}\right]\{10\,t\}\\ &= -\frac{10}{4}\,t\ ;\\ U(t) &= S_{comp}(t) + I_{pi}(t) = \sigma_0\exp(2\,t) + \sigma_1\exp(-2\,t) - \frac{10}{4}\,t\ ;\\ U(t) \to u(x) &= \sigma_0(2x+3) + \frac{\sigma_1}{(2x+3)} - \frac{5}{4}\log(2x+3). \end{aligned} \tag{6.197}$$

$$X(B): \ \left[(x+2)^2 D^2 + 3(x+2) D + 1\right] u(x) = (x+2)^2 \log(x+2) \tag{6.198}$$

Solution of Example X(B)

According to the schedule described earlier, set $(x+2) = \exp(t)$ and $(x+2)D = \Delta$ and transform (6.198) into the following:

$$\begin{aligned} [\Delta(\Delta-1) + 3\Delta + 1]\, U(t) &= \exp(2t) \log\left[\exp(t)\right] \\ = \left[\Delta^2 + 2\Delta + 1\right] U(t) \quad &= \exp(2t)\, t. \end{aligned} \tag{6.199}$$

Now proceed with the usual routine. That is

$$\begin{aligned} E_{ch} &= k^2 + 2k + 1 = 0 \quad ; \ k_{1,2} = -1 \ ; \\ S_{comp}(t) &= (\sigma_0 + \sigma_1 t) \exp(-t) \ . \\ I_{pi}(t) &= \frac{1}{\left[\Delta^2 + 2\Delta + 1\right]} \{\exp(2t)\, t\} = \exp(2t) \frac{1}{\left[(\Delta+2)^2 + 2(\Delta+2) + 1\right]} \{t\} \\ &= \exp(2t) \frac{1}{\left[9 + 6\Delta + \Delta^2\right]} \{t\} = \frac{\exp(2t)}{9} \left(1 - \frac{6}{9}\Delta\right) t \\ = I_{pi}(t) &= \frac{\exp(2t)}{9} \left(t - \frac{6}{9}\right) \quad ; U(t) = S_{comp}(t) + I_{pi}(t) \\ &= (\sigma_0 + \sigma_1 t) \exp(-t) + \frac{\exp(2t)}{9} \left(t - \frac{6}{9}\right) \ . \\ U(t) \ \rightarrow \ u(x) &= \frac{\sigma_0 + \sigma_1 \log(x+2)}{(x+2)} + \frac{(x+2)^2}{9} \left[\log(x+2) - \frac{6}{9}\right]. \end{aligned} \tag{6.200}$$

$$X(C): \ \left[(x-1)^2 D^2 - 1\right] u(x) = (x-1) \cos[\log(x-1)] \log(x-1) \tag{6.201}$$

Solution of Example X(C)

According to the schedule described earlier, set $(x-1) = \exp(t)$, $(x-1)^2 D^2 = \Delta(\Delta-1)$ and transform (6.201) into the following:

$$\begin{aligned} [\Delta(\Delta-1) - 1]\, U(t) &= \exp(t)\ \cos(t)\, t \\ = \left[\Delta^2 - \Delta - 1\right] U(t) \quad &= \exp(t)\ \cos(t)\, t. \end{aligned} \tag{6.202}$$

Again proceed with the usual routine.

$$E_{ch} = k^2 - k - 1 = 0 \; ; \; k_{1,2} = \frac{1 \pm \sqrt{5}}{2} \; ;$$

$$S_{comp}(t) = \exp\left(\frac{t}{2}\right)\left\{\sigma_0 \exp\left(\frac{-\sqrt{5}}{2}\,t\right) + \sigma_1 \exp\left(\frac{\sqrt{5}}{2}\,t\right)\right\};$$

$$I_{pi}(t) = \frac{1}{\left[\Delta^2 - \Delta - 1\right]}\{\exp(t)\,\cos(t\,t\} = \frac{1}{\left[\Delta^2 - \Delta - 1\right]}\{\exp(t)\frac{\exp(i\,t) + \exp(-i\,t)}{2}t\}.$$

$$= \exp[t(1+i)]\frac{1}{\left[(\Delta + 1 + i)^2 - (\Delta + 1 + i) - 1\right]}\{t\}$$

$$+ \exp[t(1-i)]\frac{1}{\left[(\Delta + 1 - i)^2 - (\Delta + 1 - i) - 1\right]}\{t\}.$$

$$= -\left(\frac{1}{5}\right)\exp[t(1+i)]\,[(2+i) - (1-2\,i)\Delta]\,\{t\}$$

$$-\left(\frac{1}{5}\right)\exp[t(1-i)]\,[(2-i) - (1+2\,i)\Delta]\,\{t\}.$$

$$= -\left(\frac{1}{5}\right)\exp[t(1+i)]\,[(2+i)t - (1-2\,i)]$$

$$-\left(\frac{1}{5}\right)\exp[t(1-i)]\,[(2-i)t - (1+2\,i)]\,.$$

$$= I_{pi}(t) = \frac{\exp(t)}{5}\,[(2+t)\sin(t) + (1-2t)\cos(t)]\,;$$

$$\begin{aligned} U(t) &= S_{comp}(t) + I_{pi}(t) \\ &= \exp\left(\frac{t}{2}\right)\left\{\sigma_0 \exp\left(\frac{-\sqrt{5}}{2}\,t\right) + \sigma_1 \exp\left(\frac{\sqrt{5}}{2}\,t\right)\right\} \\ &+ \frac{\exp(t)}{5}\,[(2+t)\sin(t) + (1-2t)\cos(t)]\;; \end{aligned}$$

$$\begin{aligned} U(t) \;\rightarrow\; u(x) &= (x-1)^{\frac{1}{2}}\left\{\sigma_0(x-1)^{\frac{-\sqrt{5}}{2}} + \sigma_1(x-1)^{\frac{\sqrt{5}}{2}}\right\} + \\ &\frac{1}{5}(x-1)\,\{2 + \log(x-1)\}\sin[\log(x-1)] + \\ &+ \frac{1}{5}(x-1)\,\{1 - 2\log(x-1)\}\cos[\log(x-1)]\;. \end{aligned} \tag{6.203}$$

6.9.4 Problems Group IX

Solve the following five problems by transforming each of the differential equations by setting the variable $(a\,x + b) = \exp(a\,t)$. Then, work out their solution $U(t)$. Also write the corresponding solution $u(x)$.

$$\left[(x+3)^2 D^2 + 2(x+3)D\right] u = -\log(x+3)\,(x+3)\ . \quad (1)$$

$$\left[(x+3)^2 D^2 + 2(x+3)D + 3\right] u = -\log(x+3)\,(x+3)\ . \quad (2)$$

$$\left[(2x-1)^2 D^2 - 2(2x-1)D\right] u = -2\,\log(2x-1)\,\sin[\log(2x-1)]\,(2x-1)\ . \quad (3)$$

$$\left[(2x-1)^2 D^2 - 2(2x-1)D + 4\right] u = -2\,\log(2x-1)\,\sin[\log(2x-1)]\,(2x-1)\ . \quad (4)$$

$$\left[(3x+2)^2 D^2 - 3(3x+2)D + \frac{1}{4}\right] u = \sin[\log(3x+2)]\,(3x+2)\ . \quad (5)$$

6.10 Factorable Equations

Occasionally, there is a second- or higher-order linear (ODE) that, by good luck or fortunate insight, can be factorized. Of course, the factorization has to be done in such a manner that it keeps the differential equation operationally intact. For instance, construct a second-order linear (ODE) with variable coefficients that is arranged in advance to be operationally factorable into the following pair of first-order equations.

$$y_1 \ = \ [D + M_2]y(x), \tag{6.204}$$

and

$$[x^n\, D + M_1]y_1 \ = \ N \ , \tag{6.205}$$

where for brevity $M_1(x)$ has been represented as M_1, $M_2(x)$ as M_2, $N(x)$ as N, $y_1(x)$ as y_1. With the help of (6.204), which allows replacing y_1 by $[D + M_2]y(x)$, (6.205) can be represented as

$$[x^n\, D + M_1][D + M_2]y(x) = N. \tag{6.206}$$

The operator multiplication of the term on the left-hand side of (6.206) requires some care. For instance, (6.206) leads to the following second-order (linODE) with variable coefficients.

$$\begin{aligned}&\left\{x^n\, D^2 + [M_1 + x^n\, M_2]\, D + [x^n\, (DM_2) + M_1 M_2]\right\} y(x)\\ &= \ N.\end{aligned} \tag{6.207}$$

In this chapter, for convenience, (6.204)–(6.207) are referred to as the 'master equations.' Analyses of factorable second-order differential equations will make extensive use of these master equations.

Unfortunately by comparing a second-order differential equation with its master equation equivalent (6.207), one can glean only limited information. For instance, while one can readily find what n and N are, determining M_1 and M_2 is less straightforward. Here, one knows only the sums $[M_1 + x^n\, M_2]$ and $[x^n\, (DM_2) + M_1 M_2]$. But alas that does not a readily accessible solution of M_1 and M_2 make.

6.10.1 Examples Group XI

Check whether the following second-order equations are factorable. And if so, solve them accordingly.

$$\begin{aligned}
XI(1) &: \left[D^2 + (x^{-1} + 1)\, D + x^{-1}\right] y(x) \;=\; x\,. \\
XI(2) &: \left[D^2 + (x^{-1} + 3)\, D + 2(x^{-1} + 1)\right] y \;=\; x^{-1}\,. \\
XI(3) &: \left[x\, D^2 - (2\,x + 1) D + 2\right] y \;=\; x^3\, \exp(x)\,. \\
XI(4) &: \left[D^2 + x^{-1}\, D - x^{-2}\right] y \;=\; \exp(x)\,. \\
XI(5) &: \left[D^2 + 2x\, D + 2\right] y \;=\; \log(x)\,. \\
XI(6) &: \left[D^2 + \exp(x)\, D + \exp(x)\right] y \;=\; 2\,x\, \exp(x)\,.
\end{aligned} \tag{6.208}$$

Solution XI(1)

One-to-one comparison with (6.207) leads to the equalities

$$M_1 + x^n\, M_2 = x^{-1} + 1\;; \tag{6.209}$$

$$x^n\, (DM_2) + M_1 M_2 = x^{-1}\;. \tag{6.210}$$

Because $N = x$ and $n = 0$, M_1 and M_2 are determined by the relations

$$M_1 + M_2 = x^{-1} + 1. \tag{6.211}$$

$$(DM_2) + M_1 M_2 = x^{-1}\;. \tag{6.212}$$

Elimination of M_1, (6.211) and (6.212), renders (6.212) into a Riccati equation—compare (6.130)–(6.156)—

$$[D + (x^{-1} + 1)] M_2 = [M_2]^2 + x^{-1}. \tag{6.213}$$

and solving it would involve a second-order differential equation. Thus, one would have made no headway at all! Unless, of course, one were clever and could guess M_1 or M_2 by inspection.

Another two possibilities for success are if
(A): M_1 is vanishing
(B): or, M_2 is a constant.

In order to solve (6.211) and (6.212), try the first possibility: meaning $M_1 = 0$. If this is true, then according to (6.211) and (6.212), $M_2 = (x^{-1} + 1)$ and $DM_2 = x^{-1}$. Which of course is not possible.

On the other hand, the second possibility, namely M_2 is a constant $= c$, works. Here, $DM_2 = 0$ and (6.213) becomes

$$(x^{-1} + 1)c = c^2 + x^{-1} \quad . \tag{6.214}$$

This is a quadratic with two solutions $c = 1$ and $c = x^{-1}$. Clearly the only acceptable solution, where $M_2 = c$ is a constant, is $c = 1$. Knowing $M_2 = 1$, (6.211), or (6.212), gives $M_1 = x^{-1}$.

Also because $n = 0$ and $N = x$, the central differential equations to be solved—namely (6.205) ≡ (a) and (6.204)≡ (b)—can now be written as:

$$\begin{aligned} (a) \ : \quad & [D + x^{-1}]y_1 = x; \\ (b) \ : \quad & [D + 1]\, y(x) \ = \ y_1 \ . \end{aligned} \tag{6.215}$$

Because both these equations are first-order (linODE), the procedures used in (4.2)–(4.18) apply and lead readily to the relevant solution.

$$\begin{aligned} &(a) \ : y_1 = \frac{\sigma_1}{x} + \frac{x^2}{3}; \\ &(b) \ : y(x) = \sigma_1 \exp(-x) \int \frac{\exp(x)}{x} \mathrm{d}x + \sigma_2 \exp(-x) + \frac{1}{3}[x^2 - 2x + 2]. \end{aligned} \tag{6.216}$$

Solution XI(2)

Again work with the master equation (6.207) and the defining (6.204), (6.205), and (6.206).

Here, $n = 0$, $N = x^{-1}$ and

$$\{2\} \ : \ (M_1 + M_2) \ = \ (x^{-1} + 3) \ ; \ [DM_2 + M_1 M_2] \ = \ 2(x^{-1} + 1). \tag{6.217}$$

Clearly, $M_1 = 0$ does not work.

Next, try $M_2 = c$. Then $DM_2 \ = \ 0$. One of the two solutions of the resultant quadratic in c is a constant equal to 2. This means $c \ = \ M_2 \ = \ 2$ and therefore $M_1 \ = \ (1 + x^{-1})$. Now that one knows all the relevant parameters, one can construct the pair of first-order (linODE) that need solving. These are:

$$\begin{aligned} (a)\ :\quad & [D + (1 + x^{-1})]y_1 \ = \ x^{-1}\ ; \\ (b)\ :\quad & (D + 2)y(x) \ = \ y_1\ . \end{aligned} \tag{6.218}$$

In the usual manner, their solution is found to be the following:

$$\begin{aligned} (a)\ :\quad & y_1 \ = \ \sigma_1 \frac{\exp(-x)}{x} + \frac{1}{x}\ ; \\ (b)\ :\quad & y(x) \ = \sigma_1 \exp(-2x) \int \left[\frac{\exp(x)}{x}\right] \mathrm{d}x + \sigma_2 \exp(-2x) \\ & \quad + \exp(-2x) \int \left[\frac{\exp(2x)}{x}\right] \mathrm{d}x. \end{aligned} \tag{6.219}$$

Solution XI(3)

Next, solve differential equation {3}. Here, $n = 1$ and

$$\{3\}\ :\ (M_1 + x\, M_2) = -(2x + 1)\ ;\ [x(DM_2) + M_1 M_2] = 2\ . \tag{6.220}$$

Therefore, $M_1 = 0$ does not work.

Next, try $M_2 = c$. Then, $DM_2 \ = \ 0$ and (6.220) represent two relations: $(M_1 + x\, c) \ = \ -\,(2x + 1)$ and $c\, M_1 \ = \ 2$. Elimination of M_1 leads to a quadratic in c

$$xc^2 + (2x + 1)c + 2 = 0. \tag{6.221}$$

Its two solutions are : $c = -x^{-1}$ and $c = -2$. Clearly, the solution which is relevant is when c is a constant. That is, $c = -2$. Knowing $M_2 = c = -2$ one readily finds $M_1 = -1$. Knowing n, M_1, M_2, and the fact that $N = x^3 \exp(x)$, one can construct the pair of first-order (linODE) that need solving. These are:

$$\begin{aligned} (a)\ :\quad & [x\, D - 1]y_1 \ = \ x^3 \exp(x); \\ (b)\ :\quad & (D - 2)y(x) \ = \ y_1\ . \end{aligned} \tag{6.222}$$

In the usual manner, their solution is found to be the following:

$$\begin{aligned} (a)\ :\quad & y_1 \ = \ \sigma_1\, x + (x^2 - x)\exp(x)\ ; \\ (b)\ :\quad & y(x) \ = \sigma_2 \exp(2x) - \left(\frac{\sigma_1}{4}\right)(2x + 1) \\ & \quad - \left(x^2 + x + 1\right)\exp(x)\ . \end{aligned} \tag{6.223}$$

Solution XI(4)

Here, $n = 0$, $N = \exp(x)$ and

$$\{4\} \ : \ (M_1 + M_2) \ = \ x^{-1} \ ; \ [DM_2 + M_1 M_2] \ = \ -x^{-2}. \tag{6.224}$$

As usual, first try $M_1 = 0$. Therefore, $M_2 \ = \ x^{-1}$. Then, $DM_2 \ = \ -x^{-2}$ and both the equations work. Now, one can readily construct the relevant pair of first-order (linODE). These are:

$$\begin{aligned} &(a) \ : \qquad Dy_1 \ = \ \exp(x) \ ; \\ &(b) \ : \qquad (D + x^{-1}) y(x) \ = \ y_1 \ . \end{aligned} \tag{6.225}$$

Their solution is

$$\begin{aligned} &(a) \ : \quad y_1 \ = \ \sigma_1 \ + \exp(x) \ ; \\ &(b) \ : \quad y(x) \ = \sigma_1 \left(\frac{x}{2}\right) + \frac{\sigma_2}{x} \\ &\qquad\qquad + \ \exp(x) - \left(\frac{1}{x}\right) \exp(x). \end{aligned} \tag{6.226}$$

Solution XI(5)

Here, $n = 0$, $N = \log(x)$ and

$$\{5\} \ : \ (M_1 + M_2) \ = \ 2x \ ; \ [DM_2 + M_1 M_2] \ = \ 2. \tag{6.227}$$

Set $M_1 = 0$. Then, $M_2 \ = \ 2x$ and $DM_2 \ = \ 2$. These two statements are consistent. Now, one can readily construct the relevant pair of first-order (linODE). These are:

$$\begin{aligned} &(a) \ : Dy_1 \ = \ \log(x); \\ &(b) \ : (D + 2x) y(x) = y_1 \ . \end{aligned} \tag{6.228}$$

Their solution is

$$\begin{aligned} &(a) \ : y_1 \ = \ \sigma_1 \ - x \ + \ x \log(x); \\ &(b) \ : y(x) \ = \sigma_1 \int \exp(x^2) \mathrm{d}x + \sigma_2 \exp(-x^2) \\ &\qquad + \ \left(\frac{1}{2}\right) [\log(x) - 1] - \left(\frac{1}{4}\right) \exp(-x^2) \int \left[\frac{\exp(x^2)}{x}\right] \mathrm{d}x. \end{aligned} \tag{6.229}$$

Solution XI(6)

Here, $n = 0$, $N = 2x \exp(x)$ and

$$\{6\} \ : \ (M_1 + M_2) \ = \ \exp(x) \ ; \ [DM_2 + M_1 M_2] \ = \ \exp(x). \tag{6.230}$$

Again, first try $M_1 = 0$. Then, $M_2 = \exp(x)$ and $DM_2 = \exp(x)$. These two statements are consistent. Therefore, the relevant pair of first-order linear (ODE) is

$$\begin{aligned} (a) \; : \quad & Dy_1 = 2x \exp(x) \; ; \\ (b) \; : \quad & \left[D + \exp(x)\right] y(x) = y_1 \; . \end{aligned} \tag{6.231}$$

Their solution is

$$\begin{aligned} (a) \; : \quad & y_1 = \sigma_1 + 2 \exp(x)\,(x-1); \\ (b) \; : \quad & y(x) = (\sigma_1 - 2) \exp\{-\exp(x)\} \int \exp\{\exp(x)\} \mathrm{d}x \\ & + \sigma_2 \exp\{-\exp(x)\} + 2x - 2 \; . \end{aligned} \tag{6.232}$$

$$u = 1 + 2y \; ; \; v = 1 + 3x. \left(\frac{\partial u}{\partial y}\right)_x = 2 \; and \; \left(\frac{\partial v}{\partial x}\right)_y = 3 \; . \tag{6.233}$$

Because $2 \neq 3$, this equation is inexact. Therefore, try using (6.99) to check as to whether an integrating factor that depends only on the variable x is possible.

$$\frac{\left[\left(\frac{\partial u}{\partial y}\right)_x - \left(\frac{\partial v}{\partial x}\right)_y\right]}{v} . \mathrm{d}x = \left[\frac{-\mathrm{d}x}{1+3x}\right] = \left[\frac{\mathrm{d}X(x)}{X(x)}\right] \; . \tag{6.234}$$

Good news! There exists an integrating factor $X(x)$. One can find it by integrating the above equation.

$$\log[X(x)] = \log\,(1+3x)^{\frac{-1}{3}} + constant. \tag{6.235}$$

Thus

$$X(x) = \frac{\sigma_0}{(1+3x)^{\frac{1}{3}}} \; . \tag{6.236}$$

Including the integrating factor, and ignoring the unnecessary multiplier σ_0, the differential equation (6.104)-(1) now is

$$(1+3x)^{-\frac{1}{3}} \left[(1+2y)\; \mathrm{d}x + (1+3x)\; \mathrm{d}y\right] = 0 \; . \tag{6.237}$$

Now determine whether this new differential equation is indeed exact. One has

$$\begin{aligned} & u(x,y) = (1+2y)(1+3x)^{-\frac{1}{3}} \; ; \; v(x,y) = (1+3x)^{\frac{2}{3}}. \; Therefore, \\ & \left(\frac{\partial u}{\partial y}\right)_x = 2(1+3x)^{-\frac{1}{3}} \; ; \; \left(\frac{\partial v}{\partial x}\right)_u = \frac{2}{3}\left[(1+3x)^{-\frac{1}{3}}\right] 3 \; . \end{aligned} \tag{6.238}$$

Good! The new equation is exact. Accordingly, one can find its solution via the procedure outlined in (6.88). That is

$$\begin{aligned}\sigma_1 - \sigma_2 &= \int u(x,y)\,\mathrm{d}x \;+\; \int v(x,y)\,\mathrm{d}y - \int \left(\frac{\partial[\int u(x,y)\,\mathrm{d}x]}{\partial y}\right)_x \mathrm{d}y \\ &= \int (1+3x)^{-\frac{1}{3}}(1+2y)\mathrm{d}x + \int (1+3x)^{\frac{2}{3}}\mathrm{d}y \\ &\quad - \int \left(\frac{\partial[\int u(x,y)\,\mathrm{d}x]}{\partial y}\right)_x \mathrm{d}y \\ &= \frac{(1+2y)(1+3x)^{\frac{2}{3}}}{\left[3\left(\frac{2}{3}\right)\right]} + (1+3x)^{\frac{2}{3}}y - (1+3x)^{\frac{2}{3}}y \\ &= \frac{(1+2y)(1+3x)^{\frac{2}{3}}}{2}. \end{aligned} \tag{6.239}$$

Therefore, the solution to (6.104)-(1) is

$$y(x) \;=\; \frac{-1}{2} + \frac{\sigma_1 - \sigma_2}{(1+3x)^{\frac{2}{3}}} \;. \tag{6.240}$$

6.10.2 Problems Group VII

Solve the following three differential equations.

[Hint: Compare (6.81)–(6.128).]

$$\begin{aligned}(1):\; 0 \;&=\; \mathrm{d}z = (2+3y)\;\mathrm{d}x + (3+2x)\;\mathrm{d}y. \\ (2):\; 0 \;&=\; \mathrm{d}z = y^3\,\mathrm{d}x + 2xy^2\,\mathrm{d}y. \\ (3):\; 0 \;&=\; \mathrm{d}z = \left(x^2+y\right)\,\mathrm{d}x + \left(\frac{1}{3x}\right)\,\mathrm{d}y. \end{aligned} \tag{6.241}$$

6.11 Additional Riccati Equations

A nonlinear differential equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 + f_1(x)\,y + f_2(x) \;=\; 0 \tag{6.242}$$

is sometime called a Riccati equation.

6.11.1 Solution

Set

$$\frac{1}{z}\frac{\mathrm{d}z}{\mathrm{d}x} = y \ . \tag{6.243}$$

As a result,

$$\frac{\mathrm{d}z}{\mathrm{d}x} = z\,y \ . \tag{6.244}$$

Its differentiation gives

$$\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} = z\frac{\mathrm{d}y}{\mathrm{d}x} + y\frac{\mathrm{d}z}{\mathrm{d}x} = z\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 z \ . \tag{6.245}$$

Now, multiply (6.242) by z

$$z\left[\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 + f_1(x)\,y + f_2(x)\right] = 0 \tag{6.246}$$

and replace $z\,\frac{\mathrm{d}y}{\mathrm{d}x}$ according to (6.245). This gives

$$\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} - y^2 z + z\,y^2 + f_1(x)\,z\,y + f_2(x)\,z = 0 \ . \tag{6.247}$$

Remove $-y^2 z + z\,y^2$ and replace $z\,y$ by $\frac{\mathrm{d}z}{\mathrm{d}x}$. As a result, the Riccati equation (6.242) is transformed into

$$\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} + f_1(x)\frac{\mathrm{d}z}{\mathrm{d}x} + f_2(x)z = 0 \ . \tag{6.248}$$

Notice that this differential equation, unlike the original Riccati equation, is linear. But the removal of nonlinearity has resulted in increasing the order. And we now have a second-order differential equation. But, because the new equation is linear, there is a greater possibility of making progress. In particular, if $f_1(x)$ and $f_2(x)$ should both turn out to be constants, (6.248) can be solved much like equations of the type (3.1), (3.5), etc. And once z has been determined, y can be calculated by a simple differentiation as is implicit in the form of (6.243).

6.11.2 Additional Examples Group VIII

First convert the Riccati equations given below into second-order linear equations and then solve them.

$$\begin{aligned}
(1): &\quad \tfrac{\mathrm{d}y}{\mathrm{d}x} + y^2 + x\,y = 0.\\
(2): &\quad \tfrac{\mathrm{d}y}{\mathrm{d}x} + y^2 + x^2\,y = 0.\\
(3): &\quad \tfrac{\mathrm{d}y}{\mathrm{d}x} + y^2 + y + 1 = 0.\\
(4): &\quad \tfrac{\mathrm{d}y}{\mathrm{d}x} + y^2 + 2\,y + 2 = 0.
\end{aligned} \tag{6.249}$$

Converting the Riccati equations into second-order linear ordinary differential equations is straightforward. All one needs to do is use (6.242) and (6.248). Accordingly, (6.249) get transformed as follows:

$$\begin{aligned}
(1) \to (1'): &\quad \tfrac{\mathrm{d}^2z}{\mathrm{d}x^2} + x\,\tfrac{\mathrm{d}z}{\mathrm{d}x} = 0\\
(2) \to (2'): &\quad \tfrac{\mathrm{d}^2z}{\mathrm{d}x^2} + x^2\,\tfrac{\mathrm{d}z}{\mathrm{d}x} = 0\\
(3) \to (3'): &\quad \tfrac{\mathrm{d}^2z}{\mathrm{d}x^2} + \tfrac{\mathrm{d}z}{\mathrm{d}x} + z = 0\\
(4) \to (4'): &\quad \tfrac{\mathrm{d}^2z}{\mathrm{d}x^2} + 2\,\tfrac{\mathrm{d}z}{\mathrm{d}x} + 2\,z = 0
\end{aligned} \tag{6.250}$$

6.11.3 Solution

Equations (6.249)-(1) **and** (6.250)-(1')

Introduce the notation $p = \frac{\mathrm{d}z}{\mathrm{d}x}$ and write (6.250)-(1') as

$$\frac{\mathrm{d}p}{\mathrm{d}x} + x\,p = 0. \tag{6.251}$$

Equation (6.251) is readily solved. [See, for instance, (4.2), (4.10) and (4.15) that describe how such first-order linear ordinary differential equations are solved.] That is

$$p = \sigma_0 \exp\left(-\frac{x^2}{2}\right). \tag{6.252}$$

Because $p = \frac{\mathrm{d}z}{\mathrm{d}x}$, its integration leads to z.

$$\int \frac{\mathrm{d}z}{\mathrm{d}x}\,\mathrm{d}x = z = \int p\,\mathrm{d}x = \sigma_0 \int \exp\left(-\frac{x^2}{2}\right)\mathrm{d}x + \sigma_1\ . \tag{6.253}$$

And once z is known, y can be determined by using (6.243). That is

$$y = \frac{1}{z}\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{p}{z} = \frac{\sigma_0 \exp\left(-\frac{x^2}{2}\right)}{\sigma_0 \int \exp\left(-\frac{x^2}{2}\right)\mathrm{d}x + \sigma_1}$$
$$= \frac{\exp\left(-\frac{x^2}{2}\right)}{\int \exp\left(-\frac{x^2}{2}\right)\mathrm{d}x + \sigma_2} \ . \tag{6.254}$$

Equations (6.249)-(2) **and** (6.250)-(2')

Follow the same procedure as in (6.251)–(6.254) and write (6.250)-(2') as

$$\frac{\mathrm{d}p}{\mathrm{d}x} + x^2\, p \ = \ 0 \ . \tag{6.255}$$

Again with the help of (4.2), (4.10), and (4.15) one gets

$$p = \sigma_0 \exp\left(-\frac{x^3}{3}\right). \tag{6.256}$$

Because $p = \frac{\mathrm{d}z}{\mathrm{d}x}$ its integration leads to z.

$$\int \mathrm{d}z = z = \sigma_0 \int \exp\left(-\frac{x^3}{3}\right)\mathrm{d}x + \sigma_1. \tag{6.257}$$

And once z is known, y can be determined by using (6.243). Much like (6.254) one can write y as

$$y = \frac{\exp\left(-\frac{x^3}{3}\right)}{\int \exp\left(-\frac{x^3}{3}\right)\mathrm{d}x + \sigma_2}. \tag{6.258}$$

Equations (6.249)-(3) **and** (6.250)-(3')

Notice that (6.250)-(3') is homogeneous linear ordinary differential equation with constant coefficients. And according to (3.13), the complimentary solution of a second-order homogeneous linear ordinary differential equation with constant coefficients can be written as

$$z \ \equiv \ S_{comp}(x) = \sigma_1 \ \exp\left(k_1\, x\right) + \sigma_2 \ \exp\left(k_2\, x\right). \tag{6.259}$$

Here, $k = k_1$ and $k = k_2$ are solutions of the characteristic equation E_{ch} for (6.250)-(3'). The procedure for finding E_{ch} is to replace z by 1 , $\frac{dz}{dx}$ by k , and $\frac{d^2z}{dz^2}$ by k^2 . Thus, the complementary solution for (6.250)-(3') is found by solving the following E_{ch}.

$$k^2 + k + 1 = 0 \tag{6.260}$$

which leads to

$$\begin{aligned} k = k_1 &= r + im = \frac{-1 + i\sqrt{3}}{2}, \\ k = k_2 &= r - im = \frac{-1 - i\sqrt{3}}{2}. \end{aligned} \tag{6.261}$$

Therefore, using (6.259)–(6.261), the result for z, according to (3.21) and (3.22), can be written as

$$\begin{aligned} z &= \sigma_1 \exp(r - i\,m)\,x + \sigma_2 \exp(r + i\,m)x \\ &= \exp(r\,x)\left[\sigma_1 \exp(-i\,m\,x) + \sigma_2 \exp(i\,m\,x)\right] \\ &= \exp(r\,x)\left[\sigma_3 \sin(m\,x) + \sigma_4 \cos(m\,x)\right] \\ &= \exp\left(-\frac{x}{2}\right)\left[\sigma_3 \sin\left(\frac{\sqrt{3}x}{2}\right) + \sigma_4 \cos\left(\frac{\sqrt{3}x}{2}\right)\right]. \end{aligned} \tag{6.262}$$

Or, alternatively, as

$$z = \sigma_0 \exp\left(-\frac{x}{2}\right)\cos\left(\sigma_5 - \frac{\sqrt{3}}{2}x\right) \quad . \tag{6.263}$$

Note, one went from (6.262)–(6.263) by introducing two new arbitrary constants σ_0 and σ_5 such that $\sigma_3 = \sigma_0 \sin(\sigma_5)$ and $\sigma_4 = \sigma_0 \cos(\sigma_5)$. This is perfectly alright as long as $\sigma_0^2 = \sigma_3^2 + \sigma_4^2$. And because all sigmas are arbitrary constants, this equality is trivially achieved.

Given z—see, (6.263)—the solution y is readily found.

$$y = \frac{1}{z}\frac{dz}{dx} = -\frac{1}{2} + \frac{\sqrt{3}}{2}\tan\left(\sigma_5 - \frac{\sqrt{3}}{2}x\right). \tag{6.264}$$

Equations (6.249)-(4) **and** (6.250)-(4')

Again follow a similar procedure to that used for (6.259)–(6.264) and write the E_{ch} for (6.250)-(4') as

$$k^2 + 2k + 2 = 0 \tag{6.265}$$

which leads to

$$\begin{aligned} k &= k_1 = r + im = -1 + i, \\ k &= k_2 = r - im = -1 - i. \end{aligned} \tag{6.266}$$

Therefore,

$$\begin{aligned} z &= \exp(-x)\,[\sigma_3 \sin x + \sigma_4 \cos x] \\ &\equiv \sigma_0 \exp(-x) \cos(\sigma_5 - x)\,. \end{aligned} \tag{6.267}$$

And the solution y is

$$y = \frac{1}{z}\frac{\mathrm{d}z}{\mathrm{d}x} = -1 + \tan(\sigma_5 - x). \tag{6.268}$$

6.11.4 Additional Problems Group VIII

Solve the following two problems that involve Riccati equations.
[Hint: Read (6.249)–(6.268).]

$$(1): \qquad \frac{\mathrm{d}y}{\mathrm{d}x} + y^2 + 2\,y + 1 = 0.$$

$$(2): \qquad \frac{\mathrm{d}y}{\mathrm{d}x} + y^2 + y + 2 = 0.$$

6.12 Additional Euler Equations

Among Euler's multifarious contributions to mathematics, in particular to the theory of differential equations, is his work related to linear ordinary differential equations with constant coefficients. $He^{7\cdot}$ showed how linear ordinary differential equations with variable coefficients of the form, $c_n\,x^n$, given below

$$\sum_{n=0}^{\nu} c_n\,x^n\,D^n u = B(x), \tag{6.269}$$

could be transformed into linear ordinary differential equations with constant coefficients. The latter are much easier to solve as was demonstrated in detail in Chaps. 3 and 4.

The Euler procedure consists in arranging an appropriate change of the independent variable that transforms the operator D into a new operator Δ. One proceeds as

follows. Set

$$x \;=\; \exp(t)$$

and differentiate with respect to x.

$$1 = \exp(t)\,.\,\frac{\mathrm{d}t}{\mathrm{d}x}. \tag{6.270}$$

Consider an arbitrary function $U(x,t)$ that is differentiable with respect to x and t. Multiply both sides of (6.270) from the right by $\frac{\mathrm{d}\,U(x,t)}{\mathrm{d}\,t}$.

$$\begin{aligned} 1 \cdot \frac{\mathrm{d}\,U(x,t)}{\mathrm{d}\,t} &= \exp(t)\,.\,\frac{\mathrm{d}t}{\mathrm{d}x}.\frac{\mathrm{d}U(x,t)}{\mathrm{d}\,t} \\ &= \exp(t) \cdot \frac{\mathrm{d}\,U(x,t)}{\mathrm{d}\,x} \\ = \frac{\mathrm{d}\,U(x,t)}{\mathrm{d}\,t} &= x\;\frac{\mathrm{d}\,U(x,t)}{\mathrm{d}\,x}. \end{aligned} \tag{6.271}$$

Because $U(x,t)$ is arbitrary, (6.271) implies $x\;\frac{\mathrm{d}}{\mathrm{d}\,x} \;=\; \frac{\mathrm{d}}{\mathrm{d}t}$: meaning

$$x\;D = \Delta. \tag{6.272}$$

Next, let us look at $x\;D\;(x\;D)$.

$$\begin{aligned} x\,D\,(x\,D) &= x\;\,(D\,x)\,D + x^2\,D^2 \\ = \quad \Delta\,(\Delta) &= x\,D + x^2\,D^2 = \Delta + x^2\,D^2. \end{aligned} \tag{6.273}$$

Therefore

$$x^2\,D^2 \equiv\; \Delta\,(\Delta - 1)\,. \tag{6.274}$$

Similarly

$$x\,D\left[x^2\,D^2\right] = \Delta\,[\Delta\,(\Delta - 1)]\,. \tag{6.275}$$

One also has

$$\begin{aligned} x\,D\left[x^2\,D^2\right] \;&=\; x\left(2\,x\,D^2\right) + x^3\,D^3 = 2\,x^2\,D^2 + x^3\,D^3 \\ = 2\,\Delta\,(\Delta - 1) + x^3\,D^3. \end{aligned} \tag{6.276}$$

The left-hand sides of (6.275) and (6.276) are equal. The equality of their right-hand sides yields

$$2\,\Delta\,(\Delta-1)+x^3\,D^3=\Delta\,[\Delta\,(\Delta-1)]\,, \tag{6.277}$$

or

$$x^3\,D^3\equiv\Delta(\Delta-1)(\Delta-2). \tag{6.278}$$

In order to proceed to the next order, multiply $x^3\,D^3$ from the left by $x\,D$ and use (6.278) to appropriately replace one of the expressions equal to $3\,x^3\,D^3$.

$$\begin{aligned} x\,D\left[x^3\,D^3\right]&=x\left(3\,x^2\,D^3\right)+x^4\,D^4=3\,x^3\,D^3+x^4\,D^4\\ &=3\,\Delta\,(\Delta-1)\;(\Delta-2)+x^4\,D^4. \end{aligned} \tag{6.279}$$

Multiply the left-hand side of (6.278) by $x\,D$ and the right-hand side by the same amount, that is Δ. One gets

$$x\,D\left[x^3\,D^3\right]=\Delta\,[\Delta(\Delta-1)(\Delta-2)]\,. \tag{6.280}$$

Now that the left-hand sides of (6.279) and (6.280) are the same, one can claim equality of their right-hand sides. That is

$$3\,\Delta\,(\Delta-1)\;(\Delta-2)+x^4\,D^4\;=\;\Delta\,[\Delta(\Delta-1)(\Delta-2)]\,,$$

which leads to the result

$$x^4\,D^4\equiv\Delta(\Delta-1)(\Delta-2)(\Delta-3). \tag{6.281}$$

Exercise

Clearly, (6.272), (6.274), (6.278), and (6.281) show a pattern. Therefore, by induction, one assumes

$$x^n\,D^n=\Delta(\Delta-1)...(\Delta-n+1). \tag{6.282}$$

Assuming (6.282) is valid when $n=\ell-1$, show that it is also valid when $n=\ell$.

6.12.1 Additional Examples IX: Euler Equation

Transform each of the four differential equations $IX-(A)\rightarrow IX-(D)$ given below by using the transformation $x=\exp(t)$, and renaming the function $u(x)$ as $U(t)$. Finally, represent the result in terms of x. Then, find $u(x)$.

$$IX-(A):\quad\left[x^2\,D^2+x\,D+1\right]u(x)=a\,x^2+b. \tag{6.283}$$

6.12.2 Solution Additional Examples IX-(A)

Use (6.270), (6.272) and (6.274) and transform (6.283) into

$$[\Delta(\Delta-1)+\Delta+1]\,U(t)=\left[\Delta^2+1\right]U(t)=a\exp(2\,t)+b. \tag{6.284}$$

Then, solve for $U(t)$ by following the familiar routine.

$$\begin{aligned}
E_{ch} &= k^2+1=0 \quad ; k_{1,2}=\pm i;\\
S_{comp}(t) &= \sigma_0\exp(-i\,t)+\sigma_i\exp(-i\,t)=\sigma_2\sin(t)+\sigma_3\cos(t);\\
I_{pi}(t) &= \frac{1}{\left[\Delta^2+1\right]}\{a\exp(2\,t)+b\}=a\exp(2\,t)\frac{1}{2^2+1}+\frac{b}{1};\\
U(t)&=S_{comp}(t)+I_{pi}(t)=\sigma_2\sin(t)+\sigma_3\cos(t)+\frac{a}{5}\exp(2\,t)+b.
\end{aligned}$$

Finally, transform $U(t)$ back into $u(x)$. In this fashion one gets:

$$U(t)\rightarrow u(x)=\sigma_2\sin[\log(x)]+\sigma_3\cos[\log(x)]+\frac{a}{5}x^2+b. \tag{6.285}$$

6.12.3 Solution Additional Examples IX-(B)

$$IX-(B):\quad \left[x^2\,D^2-2\,x\,D+4\right]u=a\log(x). \tag{6.286}$$

As above use (6.270), (6.272), and (6.274) and transform (6.286) into the following differential equation.

$$\left[\Delta^2-3\Delta+4\right]U(t)=a\,t. \tag{6.287}$$

Its solution is found in the usual manner.

$$\begin{aligned}
E_{ch} &= k^2-3\,k+4=0 \quad ; k_{1,2}=\frac{3}{2}\pm i\frac{\sqrt{7}}{2};\\
S_{comp}(t)&=\sigma_1\exp\left(\frac{3t}{2}\right)\sin\left(\frac{\sqrt{7}}{2}t\right)+\sigma_2\exp\left(\frac{3t}{2}\right)\cos\left(\frac{\sqrt{7}}{2}t\right);\\
I_{pi}(t)&=\frac{1}{\left[\Delta^2-3\Delta+4\right]}(a\,t)=\left(\frac{1}{4}+\frac{3}{16}\Delta\right)(a\,t)=\frac{a}{4}t+\frac{3a}{16};\\
U(t)&=S_{comp}(t)+I_{pi}(t) \quad ;\\
U(t)\rightarrow u(x) &= \sigma_1\sin\left[\frac{\sqrt{7}}{2}\log(x)\right]x^{\frac{3}{2}}+\sigma_2\cos\left[\frac{\sqrt{7}}{2}\log(x)\right]x^{\frac{3}{2}}
\end{aligned}$$

$$+\frac{a}{4}\log(x)+\frac{3a}{16}. \tag{6.288}$$

Solution Example IX-(C)

$$IX-(C):\left[x^2\,D^2+2x\,D-4\right]u=a\,x^2\,\log(x)+b\ . \tag{6.289}$$

Again use (6.270), (6.272), and (6.274) and transform (6.289) into

$$\left[\Delta^2+\Delta-4\right]U(t)=a\,\exp(2\,t)\,t+b, \tag{6.290}$$

Its solution is as follows:

$$\begin{aligned}
E_{ch}\ &=k^2+k-4=0 \quad ;k_{1,2}=-\frac{1}{2}\pm\frac{\sqrt{17}}{2};\\
S_{comp}(t)&=\sigma_1\exp\left(-\frac{t}{2}+\frac{\sqrt{17}}{2}t\right)+\sigma_2\exp\left(-\frac{t}{2}-\frac{\sqrt{17}}{2}t\right);\\
I_{pi}(t)&=\frac{1}{\left[\Delta^2+\Delta-4\right]}\{a\,\exp(2\,t)\,t\,+b\}=a\exp(2t)\frac{1}{\{(\Delta+2)^2+(\Delta+2)-4\}}t\\
+\left(\frac{b}{-4}\right)&=a\exp(2t)\left[\frac{1}{2+5\Delta+\Delta^2}\right]t-\frac{b}{4}=a\exp(2t)\left[\frac{1}{2}-\frac{5}{4}\Delta\right]t-\frac{b}{4};\\
I_{pi}(t)\ &=a\exp(2t)\left[\frac{1}{2}t-\frac{5}{4}\right]-\frac{b}{4};\\
U(t)\ &=S_{comp}(t)+I_{pi}(t);\\
U(t)\ \rightarrow\ u(x)\ &=\sigma_1\,x^{-\frac{1}{2}+\frac{\sqrt{17}}{2}}+\sigma_2\,x^{-\frac{1}{2}-\frac{\sqrt{17}}{2}}\\
&\quad+a\,x^2\left[\frac{1}{2}\log(x)-\frac{5}{4}\right].
\end{aligned} \tag{6.291}$$

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Solution Example IX-(D)

$$IX-(D):\left[x^2\,D^2-4x\,D+2\right]u=2\,x\,\cos\left[\log(x)\right]. \tag{6.292}$$

And finally, as before, use (6.270), (6.272), and (6.274) and transform (6.292) into

$$\left[\Delta^2-5\Delta+2\right]U(t)=2\,\exp(t)\cos(t). \tag{6.293}$$

This leads to

$$E_{ch} = k^2 - 5k + 2 \; ; \; k_{1,2} = \frac{5}{2} \pm \frac{\sqrt{17}}{2};$$

$$S_{comp}(t) = \sigma_1 \exp\left(\frac{5}{2}t + \frac{\sqrt{17}}{2}t\right) + \sigma_2 \exp\left(\frac{5}{2}t - \frac{\sqrt{17}}{2}t\right);$$

$$I_{pi}(t) = \frac{1}{\left[\Delta^2 - 5\Delta + 2\right]}\{2\,a\,\exp(t)\cos(t)\} = \frac{1}{\left[\Delta^2 - 5\Delta + 2\right]}\exp[t(1+i)]$$

$$+\frac{1}{\left[\Delta^2 - 5\Delta + 2\right]}\exp[t(1-i)];$$

$$= \exp[t(1+i)]\left[\frac{1}{-3-3i}\right] + \exp[t(1-i)]\left[\frac{1}{-3+3i}\right]$$

$$= -\frac{\exp(t)}{6}\left[(1-i)\exp(i\,t) + (1+i)\exp(-i\,t)\right] = I_{pi}(t) = -\frac{1}{3}\exp(t)\left[\sin(t) + \cos(t)\right].$$

$$\begin{aligned} U(t) &= S_{comp}(t) + I_{pi}(t) \\ U(t) &\rightarrow u(x) = \sigma_1\, x^{\frac{5}{2}+\frac{\sqrt{17}}{2}} + \sigma_2\, x^{\frac{5}{2}-\frac{\sqrt{17}}{2}} \\ &-\frac{x}{3}\left[\sin\{\log(x)\} + \cos\{\log(x)\}\right]. \end{aligned} \tag{6.294}$$

6.12.4 Additional Extended Euler Equations

A simple extension of Euler equation is given below in (6.295).

$$\sum_{n=0}^{\nu} c_n\,(a\,x+b)^n\,D^n u = B(x), \tag{6.295}$$

Any such (linODE) with variable coefficients of the form $c_n\,(a\,x+b)^n$ can readily be transformed into a (linODECC)—meaning into equations of the type that we have been studying previously. The relevant transformation is carried out by a change of the variable: from $(a\,x+b)$ to $\exp(a\,t)$ where both a and b are constants. That is

$$(a\,x+b) = \exp(a\,t). \tag{6.296}$$

Differentiate (6.296) with respect to x , and divide both sides by the constant a.

$$\begin{aligned} 1 &= \exp(a\,t)\,.\,\frac{\mathrm{d}t}{\mathrm{d}x} \\ &= (a\,x+b)\,.\,\frac{\mathrm{d}t}{\mathrm{d}x}. \end{aligned} \tag{6.297}$$

Consider an arbitrary function $U(x,t)$ that is differentiable with respect to x and t. Multiply both sides of (6.297) from the right by $\frac{\mathrm{d}U(x,t)}{\mathrm{d}t}$.

$$\frac{\mathrm{d}U(x,t)}{\mathrm{d}t} = (a\,x+b)\,.\,\frac{\mathrm{d}t}{\mathrm{d}x}\,\frac{\mathrm{d}U(x,t)}{\mathrm{d}t}$$
$$= (a\,x+b)\,.\,\frac{\mathrm{d}U(x,t)}{\mathrm{d}x}. \tag{6.298}$$

Because $U(x,t)$ is arbitrary, this implies $\frac{\mathrm{d}}{\mathrm{d}t} = (a\,x+b)\,\frac{\mathrm{d}}{\mathrm{d}x}$: meaning

$$\Delta = (a\,x+b)\,D. \tag{6.299}$$

Next, let us look at $(a\,x+b)\,D\,\{(a\,x+b)\,D\}$ or in other words at $\Delta\,\{\Delta\}$.

$$(a\,x+b)\,D\,\{(a\,x+b)\,D\} = (a\,x+b)[a\,D + (a\,x+b)\,D^2]$$
$$= \Delta\,\{\Delta\} \qquad = a\,\Delta + (a\,x+b)^2 D^2. \tag{6.300}$$

Transferring $a\,\Delta$ across to the left side gives

$$(a\,x+b)^2 D^2 = \Delta\,(\Delta - a)\,. \tag{6.301}$$

Similarly

$$(a\,x+b)\,D\left[(a\,x+b)^2\,D^2\right] = (a\,x+b)\,\left(2a^2x+2ab\right)\,D^2 + (a\,x+b)[(a\,x+b)^2\,D^3]$$
$$= 2a(ax+b)^2 D^2 + (ax+b)^3 D^3$$
$$= 2\,a\,\Delta\,(\Delta - a) + (a\,x+b)^3\,D^3. \tag{6.302}$$

By using (6.301) and the equivalence $(a\,x+b)\,D = \Delta$ on the left-hand side of (6.302), one gets

$$\Delta\,[\Delta\,(\Delta - a)] = 2\,a\,\Delta\,(\Delta - a) + (a\,x+b)^3\,D^3, \tag{6.303}$$

which leads to

$$(a\,x+b)^3\,D^3 = \Delta(\Delta - a)(\Delta - 2\,a). \tag{6.304}$$

Again, starting with $(a\,x+b)\,D\left[(a\,x+b)^3\,D^3\right]$, one can readily show that

$$(a\,x+b)^4\,D^4 = \Delta(\Delta - a)(\Delta - 2\,a)(\Delta - 3\,a). \tag{6.305}$$

Also, in view of (6.299), (6.301), (6.304), (6.305), or equivalently (6.306) given below, one can transform a general equation of the type (6.295) into one involving Δ's and functions of the variable t. One must not forget, however, to use $(ax+b) = \exp(at)$ for transforming out of the variable $B(x)$ into an appropriate function of the variable t. This will become clear in examples (XI) given below.

Exercise

Clearly, (6.299), (6.301), (6.304), and (6.305) show a pattern. Therefore, by induction

$$(a\,x+b)^n\,D^n = \Delta\{\Delta - a\}...\{\Delta-(n-1)a\}. \tag{6.306}$$

Assuming (6.306) is valid when $n = \ell - 1$, show that it is also valid when $n = \ell$.

6.12.5 *Additional Examples Group X: Extended Euler Equations*

Transform each of the following three differential equations by setting the variable $(a\,x+b) = \exp(a\,t)$, and then, find their solution $U(t)$. Also write the corresponding solution $u(x)$.

$$X(A): \ \left[(2x+3)^2\,D^2 + 2(2x+3)\,D - 4\right]u(x) = 5\log[2x+3]. \tag{6.307}$$

Solution of Example X(A)

Set $(2x+3) = \exp(2\,t)$ and $(2x+3)D = \Delta$, use (6.299) and (6.301), and transform (6.307) into the following:

$$\begin{aligned}[\Delta(\Delta-2) + 2\Delta - 4]\,U(t) &= 5\log\left[\exp(2t)\right] \\ &= \left[\Delta^2 - 4\right]U(t) = 10\,t.\end{aligned} \tag{6.308}$$

Now, proceed with the usual routine.

$$\begin{aligned} E_{ch} &= k^2 - 4 = 0 \quad ; k_{1,2} = \pm 2; \\ S_{comp}(t) &= \sigma_0\exp(2\,t) + \sigma_1\exp(-2\,t) \quad ; \\ I_{pi}(t) &= \frac{1}{\left[\Delta^2-4\right]}\{10\,t\} = -\frac{1}{4}\left[1+\frac{\Delta^2}{4}\right]\{10\,t\} \\ &= -\frac{10}{4}\,t \ ; \\ U(t) &= S_{comp}(t) + I_{pi}(t) = \sigma_0\exp(2\,t) + \sigma_1\exp(-2\,t) - \frac{10}{4}\,t \ ; \end{aligned}$$

$$U(t) \ \rightarrow \ u(x) = \sigma_0(2x+3) + \frac{\sigma_1}{(2x+3)} - \frac{5}{4}\log(2x+3). \tag{6.309}$$

$$X(B): \ \left[(x+2)^2\,D^2 + 3(x+2)\,D + 1\right]u(x) = (x+2)^2\log(x+2) \tag{6.310}$$

Solution of Example X(B)

According to the schedule described earlier, set $(x+2) = \exp(t)$ and $(x+2)D = \Delta$ and transform (6.310) into the following:

$$\begin{aligned}[\Delta(\Delta-1)+3\Delta+1]\,U(t) &= \exp(2\,t)\log\left[\exp(t)\right]\\ =\left[\Delta^2+2\Delta+1\right]U(t) \quad &= \exp(2\,t)\,t. \end{aligned} \tag{6.311}$$

Now proceed with the usual routine. That is

$$\begin{aligned} E_{ch} &= k^2+2k+1=0 \quad ; k_{1,2}=-1;\\ S_{comp}(t) &= (\sigma_0+\sigma_1\,t)\exp(-t) \quad .\\ I_{pi}(t) &= \frac{1}{\left[\Delta^2+2\Delta+1\right]}\{\exp(2\,t)\,t\} = \exp(2\,t)\frac{1}{\left[(\Delta+2)^2+2(\Delta+2)+1\right]}\{t\}\\ &= \exp(2\,t)\frac{1}{\left[9+6\Delta+\Delta^2\right]}\{t\} = \frac{\exp(2\,t)}{9}\left(1-\frac{6}{9}\Delta\right)t\\ &= I_{pi}(t) = \frac{\exp(2\,t)}{9}\left(t-\frac{6}{9}\right) \quad ; U(t) = S_{comp}(t)+I_{pi}(t)\\ &= (\sigma_0+\sigma_1\,t)\exp(-t)+\frac{\exp(2\,t)}{9}\left(t-\frac{6}{9}\right).\\ U(t)\to u(x) &= \frac{\sigma_0+\sigma_1\log(x+2)}{(x+2)}+\frac{(x+2)^2}{9}\left[\log(x+2)-\frac{6}{9}\right]. \end{aligned} \tag{6.312}$$

$$X(C): \ \left[(x-1)^2\,D^2-1\right]u(x) = (x-1)\cos[\log(x-1)]\log(x-1) \tag{6.313}$$

Solution of Example X(C)

According to the schedule described earlier, set $(x-1) = \exp(t)$, $(x-1)^2D^2 = \Delta(\Delta-1)$ and transform (6.313) into the following:

$$\begin{aligned}[\Delta(\Delta-1)-1]\,U(t) &= \exp(t)\,\cos(t)\,t\\ =\left[\Delta^2-\Delta-1\right]U(t) \quad &= \exp(t)\,\cos(t)\,t. \end{aligned} \tag{6.314}$$

Again proceed with the usual routine.

$$E_{ch} = k^2 - k - 1 = 0; k_{1,2} = \frac{1 \pm \sqrt{5}}{2};$$

$$S_{comp}(t) = \exp\left(\frac{t}{2}\right)\left\{\sigma_0 \exp\left(\frac{-\sqrt{5}}{2}\,t\right) + \sigma_1 \exp\left(\frac{\sqrt{5}}{2}\,t\right)\right\};$$

$$I_{pi}(t) = \frac{1}{\left[\Delta^2 - \Delta - 1\right]}\{\exp(t)\,\cos(t)\,t\} = \frac{1}{\left[\Delta^2 - \Delta - 1\right]}\{\exp(t)\frac{\exp(i\,t) + \exp(-i\,t)}{2}t\}.$$

$$= \exp[t(1+i)]\frac{1}{\left[(\Delta + 1 + i)^2 - (\Delta + 1 + i) - 1\right]}\{t\}$$

$$+ \exp[t(1-i)]\frac{1}{\left[(\Delta + 1 - i)^2 - (\Delta + 1 - i) - 1\right]}\{t\}.$$

$$= -\left(\frac{1}{5}\right)\exp[t(1+i)]\,[(2+i) - (1-2\,i)\Delta]\,\{t\}$$

$$-\left(\frac{1}{5}\right)\exp[t(1-i)]\,[(2-i) - (1+2\,i)\Delta]\,\{t\}.$$

$$= -\left(\frac{1}{5}\right)\exp[t(1+i)]\,[(2+i)t - (1-2\,i)]$$

$$-\left(\frac{1}{5}\right)\exp[t(1-i)]\,[(2-i)t - (1+2\,i)]\,.$$

$$= I_{pi}(t) = \frac{\exp(t)}{5}\,[(2+t)\sin(t) + (1-2t)\cos(t)]\,;$$

$$U(t) = S_{comp}(t) + I_{pi}(t)$$

$$= \exp\left(\frac{t}{2}\right)\left\{\sigma_0 \exp\left(\frac{-\sqrt{5}}{2}\,t\right) + \sigma_1 \exp\left(\frac{\sqrt{5}}{2}\,t\right)\right\}$$

$$+ \frac{\exp(t)}{5}\,[(2+t)\sin(t) + (1-2t)\cos(t)]\,;$$

$$U(t) \rightarrow u(x) = (x-1)^{\frac{1}{2}}\left\{\sigma_0(x-1)^{\frac{-\sqrt{5}}{2}} + \sigma_1(x-1)^{\frac{\sqrt{5}}{2}}\right\}$$

$$+\frac{1}{5}(x-1)\,\{2 + \log(x-1)\}\sin[\log(x-1)] +$$

$$+\frac{1}{5}(x-1)\,\{1 - 2\log(x-1)\}\cos[\log(x-1)]. \tag{6.315}$$

6.12.6 Additional Problems Group IX

Solve the following five problems by transforming each of the differential equations by setting the variable $(a\,x + b) = \exp(a\,t)$. Then, work out their solution $U(t)$. Also write the corresponding solution $u(x)$.

$$\left[(x+3)^2D^2+2(x+3)D\right]u=-\log(x+3)\,(x+3)\ .\ (1)$$

$$\left[(x+3)^2D^2+2(x+3)D+3\right]u=-\log(x+3)\,(x+3)\ .\ (2)$$

$$\left[(2x-1)^2D^2-2(2x-1)D\right]u=-2\,\log(2x-1)\,\sin[\log(2x-1)]\,(2x-1)\ .\ (3)$$

$$\left[(2x-1)^2D^2-2(2x-1)D+4\right]u=-2\,\log(2x-1)\,\sin[\log(2x-1)]\,(2x-1)\ .\ (4)$$

$$\left[(3x+2)^2D^2-3(3x+2)D+\frac{1}{4}\right]u=\sin[\log(3x+2)]\,(3x+2)\ .\ (5)$$

6.13 Additional Factorable Equations

Occasionally there is a second- or higher-order (linODE) that, by good luck or fortunate insight, can be factorized. Of course, the factorization has to be done in such a manner that it keeps the differential equation operationally intact. For instance, construct a second-order (linODE) with variable coefficients that is arranged in advance to be operationally factorable into the following pair of first-order equations.

$$y_1 \;=\; [D+M_2]y(x), \tag{6.316}$$

and

$$[x^n\,D+M_1]y_1=N, \tag{6.317}$$

where for brevity $M_1(x)$ has been represented as M_1, $M_2(x)$ as M_2, $N(x)$ as N, $y_1(x)$ as y_1. With the help of (6.316), which allows replacing y_1 by $[D+M_2]y(x)$, (6.317) can be represented as

$$[x^n\,D+M_1][D+M_2]y(x)=N. \tag{6.318}$$

The operator multiplication of the term on the left-hand side of (6.318) requires some care. For instance, (6.318) leads to the following second-order (linODE) with variable coefficients.

$$\begin{aligned}&\left\{x^n\,D^2+[M_1+x^n\,M_2]\,D+[x^n\,(DM_2)+M_1M_2]\right\}y(x)\\&=N.\end{aligned} \tag{6.319}$$

In this chapter, for convenience, (6.316)–(6.319) are referred to as the 'master equations.' Analyses of factorable second-order differential equations will make extensive use of these master equations.

Unfortunately by comparing a second-order differential equation with its master equation equivalent (6.319), one can glean only limited information. For instance,

while one can readily find what n and N are, determining M_1 and M_2 is less straightforward. Here, one knows only the sums $[M_1 + x^n\, M_2]$ and $[x^n\, (DM_2) + M_1 M_2]$. But alas that does not a readily accessible solution of M_1 and M_2 make.

6.13.1 Examples Group XI: Additional Factorable Equations

Check whether the following second-order equations are factorable. And if so, solve them accordingly.

$$\begin{aligned}
XI(1) &: \left[D^2 + (x^{-1} + 1)\, D + x^{-1}\right] y(x) = x. \\
XI(2) &: \left[D^2 + (x^{-1} + 3)\, D + 2(x^{-1} + 1)\right] y \;= x^{-1}. \\
XI(3) &: \left[x\, D^2 - (2\,x + 1) D + 2\right] y \;=\; x^3\, \exp(x). \\
XI(4) &: \left[D^2 + x^{-1}\, D - x^{-2}\right] y \;= \exp(x). \\
XI(5) &: \left[D^2 + 2x\, D + 2\right] y \;=\; \log(x). \\
XI(6) &: \left[D^2 + \exp(x)\, D + \exp(x)\right] y \;=\; 2\,x\, \exp(x).
\end{aligned} \tag{6.320}$$

Solution: Example XI(1)

One-to-one comparison with (6.319) leads to the equalities

$$M_1 + x^n\, M_2 = x^{-1} + 1 \;; \tag{6.321}$$

$$x^n\, (DM_2) + M_1 M_2 = x^{-1} \;. \tag{6.322}$$

Because $N = x$ and $n = 0$, M_1 and M_2 are determined by the relations

$$M_1 + M_2 = x^{-1} + 1. \tag{6.323}$$

$$(DM_2) + M_1 M_2 = x^{-1} \;. \tag{6.324}$$

Elimination of M_1, from (6.323) and (6.324), renders (6.324) into a Riccati equation—compare (6.242)–(6.268)—

$$[D + (x^{-1} + 1)] M_2 = [M_2]^2 + x^{-1}. \tag{6.325}$$

and solving it would involve a second-order differential equation. Thus, one would have made no headway at all ! Unless, of course, one were clever and could guess M_1 or M_2 by inspection.

Another two possibilities for success are if
(A): M_1 is vanishing
(B): or, M_2 is a constant.

In order to solve (6.323) and (6.324), try the first possibility: meaning $M_1 = 0$. If this is true, then according to (6.323) and (6.324), $M_2 = (x^{-1} + 1)$ and $DM_2 = x^{-1}$.

Which of course is not possible.
On the other hand, the second possibility, namely M_2 is a constant $= c$, works. Here, $DM_2 = 0$ and (6.325) becomes

$$(x^{-1} + 1)c = c^2 + x^{-1}. \tag{6.326}$$

This is a quadratic with two solutions $c = 1$ and $c = x^{-1}$. Clearly, the only acceptable solution, where $M_2 = c$ is a constant, is $c = 1$. Knowing $M_2 = 1$, (6.323), or (6.324), gives $M_1 = x^{-1}$.

Also because $n = 0$ and $N = x$, the central differential equations to be solved—namely (6.317)≡ (a) and (6.316)≡ (b)—can now be written as:

$$\begin{aligned} (a) \ : \quad & [D + x^{-1}]y_1 = x; \\ (b) \ : \quad & [D + 1]\, y(x) = y_1. \end{aligned} \tag{6.327}$$

Because both these equations are first-order (linODE), the procedures used in (4.2)–(4.18) apply and lead readily to the relevant solution.

$$\begin{aligned} &(a) : y_1 = \frac{\sigma_1}{x} + \frac{x^2}{3}; \\ &(b) : y(x) = \sigma_1 \exp(-x) \int \frac{\exp(x)}{x} dx + \sigma_2 \exp(-x) + \frac{1}{3}[x^2 - 2x + 2]. \end{aligned} \tag{6.328}$$

Solution: Example XI(2)

Again work with the master equation (6.319) and the defining (6.316), (6.317), and (6.318).

Here, $n = 0$, $N = x^{-1}$ and

$$\{2\} \ : (M_1 + M_2) \ = \ (x^{-1} + 3) \ ; \ [DM_2 + M_1 M_2] \ = \ 2(x^{-1} + 1). \tag{6.329}$$

Clearly, $M_1 = 0$ does not work.

Next, try $M_2 = c$. Then $DM_2 \ = \ 0$. One of the two solutions of the resultant quadratic in c is a constant equal to 2. This means $c \ = \ M_2 \ = \ 2$ and therefore $M_1 \ = \ (1 + x^{-1})$. Now that one knows all the relevant parameters, one can construct the pair of first-order (linODE) that need solving. These are:

$$\begin{aligned} (a) \ : \quad & [D + (1 + x^{-1})]y_1 \ = \ x^{-1} \ ; \\ (b) \ : \quad & (D + 2)y(x) \ = \ y_1 \ . \end{aligned} \tag{6.330}$$

In the usual manner, their solution is found to be the following:

$$\begin{aligned}(a)\ :\quad & y_1 \;=\; \sigma_1\,\frac{\exp(-x)}{x} + \frac{1}{x}\ ;\\ (b)\ :\quad & y(x) \;=\sigma_1\,\exp(-2x)\int\left[\frac{\exp(x)}{x}\right]\mathrm{d}x + \sigma_2\,\exp(-2x)\\ & \quad +\ \exp(-2x)\int\left[\frac{\exp(2x)}{x}\right]\mathrm{d}x. \end{aligned} \tag{6.331}$$

Solution: Example XI(3)

Next, solve differential equation {3}. Here, $n = 1$ and

$$\{3\}\ :\ (M_1 + x\,M_2) = -(2x+1)\ ;\ [x(DM_2) + M_1M_2] = 2\ . \tag{6.332}$$

Therefore, $M_1 = 0$ does not work.

Next, try $M_2 = c$. Then $DM_2 = 0$ and (6.332) represent two relations: $(M_1 + x\,c) = -(2x+1)$ and $c\,M_1 = 2$. Elimination of M_1 leads to a quadratic in c

$$xc^2 + (2x+1)c + 2 = 0\ . \tag{6.333}$$

Its two solutions are : $c = -x^{-1}$ and $c = -2$. Clearly, the solution which is relevant is when c is a constant. That is, $c = -2$. Knowing $M_2 = c = -2$, one readily finds $M_1 = -1$. Knowing n, M_1, M_2, and the fact that $N = x^3\exp(x)$, one can construct the pair of first-order (linODE) that need solving. These are:

$$\begin{aligned}(a)\ :\quad & [x\,D - 1]y_1 \;=\; x^3\exp(x)\ ;\\ (b)\ :\quad & (D-2)y(x) \;=\; y_1\ . \end{aligned} \tag{6.334}$$

In the usual manner, their solution is found to be the following:

$$\begin{aligned}(a)\ :\quad & y_1 \;=\; \sigma_1\,x + (x^2 - x)\exp(x)\ ;\\ (b)\ :\quad & y(x) \;=\sigma_2\,\exp(2x) - \left(\frac{\sigma_1}{4}\right)(2x+1)\\ & \quad -\left(x^2 + x + 1\right)\exp(x)\ . \end{aligned} \tag{6.335}$$

Solution: Example XI(4)

Here, $n = 0$, $N = \exp(x)$ and

$$\{4\}\ :\ (M_1 + M_2)\ =\ x^{-1}\ ;\ [DM_2 + M_1M_2]\ =\ -x^{-2}. \tag{6.336}$$

As usual, first try $M_1 = 0$. Therefore, $M_2 = x^{-1}$. Then, $DM_2 = -x^{-2}$ and both the equations work. Now, one can readily construct the relevant pair of first-order (linODE). These are:

$$\begin{aligned} (a) &: \quad Dy_1 = \exp(x) \ ; \\ (b) &: \quad (D + x^{-1})y(x) = y_1 \ . \end{aligned} \tag{6.337}$$

Their solution is

$$\begin{aligned} (a) &: \ y_1 = \sigma_1 + \exp(x) \ ; \\ (b) &: \ y(x) = \sigma_1 \left(\frac{x}{2}\right) + \frac{\sigma_2}{x} \\ &\quad + \exp(x) - \left(\frac{1}{x}\right)\exp(x). \end{aligned} \tag{6.338}$$

Solution: Example XI(5)

Here, $n = 0$, $N = \log(x)$ and

$$\{5\} \ : \ (M_1 + M_2) = 2x \ ; \ [DM_2 + M_1 M_2] = 2. \tag{6.339}$$

Set $M_1 = 0$. Then, $M_2 = 2x$ and $DM_2 = 2$. These two statements are consistent. Now, one can readily construct the relevant pair of first-order (linODE). These are:

$$\begin{aligned} (a) &: \quad Dy_1 = \log(x) \ ; \\ (b) &: \quad (D + 2x)y(x) = y_1 \ . \end{aligned} \tag{6.340}$$

Their solution is

$$\begin{aligned} (a) &: \ y_1 = \sigma_1 - x + x\log(x); \\ (b) &: \ y(x) = \sigma_1 \int \exp(x^2)\mathrm{d}x + \sigma_2 \exp(-x^2) \\ &\quad + \left(\frac{1}{2}\right)[\log(x) - 1] - \left(\frac{1}{4}\right)\exp(-x^2)\int \left[\frac{\exp(x^2)}{x}\right]\mathrm{d}x. \end{aligned} \tag{6.341}$$

Solution: Example XI(6)

Here, $n = 0$, $N = 2x\exp(x)$, and

$$\{6\} \ : \ (M_1 + M_2) = \exp(x) \ ; \ [DM_2 + M_1 M_2] = \exp(x). \tag{6.342}$$

Again, first try $M_1 = 0$. Then, $M_2 = \exp(x)$ and $DM_2 = \exp(x)$. These two statements are consistent. Therefore, the relevant pair of first-order (linODE) is

$$\begin{aligned} (a) \; : \quad & Dy_1 = 2x \exp(x) \; ; \\ (b) \; : \quad & \big[D + \exp(x)\big] y(x) = y_1 \; . \end{aligned} \tag{6.343}$$

Their solution is

$$\begin{aligned} (a) \; : \quad & y_1 = \sigma_1 + 2 \exp(x) (x - 1) \; ; \\ (b) \; : \quad & y(x) = (\sigma_1 - 2) \exp\{-\exp(x)\} \int \exp\{\exp(x)\} \mathrm{d}x \\ & + \sigma_2 \exp\{-\exp(x)\} + 2x - 2 \; . \end{aligned} \tag{6.344}$$

Chapter 7
Special Situations



Unlike equations of the first-order and first-degree linear (ODE)'s that are second and higher order, and have variable coefficients, are often difficult to solve. Fortunately, there are equations of special types that are easier to handle. Some of these were treated in the preceding chapter. Different from special types, but somewhat in the same vein, are equations that represent 'Special Situations'. With such situations in place, satisfactory solution is often possible. For instance, a given differential equation may be integrable. Similarly, equations that have both the independent as well as the dependent variables missing in any explicit form: And those that explicitly contain only the independent variable, or only the dependent variable, are easily handled.

An interesting new situation called order reduction comes into play if one of the n non-trivial solution of an nth order homogeneous linear (ODE) is already known. Then, the given equation can be reduced to one of $(n-1)$th order.

7.1 Ordinary Differential Equations

Both homogeneous—[See (3.1)–(3.56)]—and inhomogeneous—[See (3.57)–(3.91)]—first- and second-order linear ordinary differential equations that have constant coefficients were discussed in detail in Chap. 3. Indeed, that discussion would also apply to similar equations of order higher than the second. Also studied, there were simultaneous linear ordinary differential equations—[See (3.92)–(3.143)]. But owing to the complexity of this latter study, simultaneous equations involving only two or three dependent variables were treated.

Chapter 4 dealt with equations of first order and first degree with variable coefficients. Included there was a discussion of the special case of the Bernouilli equations

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[See (4.19)–(4.77)]. These equations, it turns out, can be transformed into standard linear (ODE)) of first order and first degree.

The Second Order

Unlike equations of the first order and first degree, as a rule linear (ODE)'s that are second and higher order and have variable coefficients are difficult to solve in terms of known elementary functions. Fortunately, there are exceptions to this gloomy rule. But, much like the $Bernouilli^{1.}$ equations, mostly the exceptions are equations of special types. Some examples of these are special equations that were described in the preceding chapter. Included there were the $Clairaut^{2.}$ equations—[Compare (6.2)–(6.13)]—the $Lagrange^{3.}$ equation—[Compare (6.19)–(6.31)]—the Separable Equations—[Compare (6.32)–(6.35)] and the $\frac{dy}{dx} = \Phi\left(\frac{y}{x}\right)$ equations—[Compare (6.36)–(6.73)]. In addition, there are the so-called Exact- [Compare (6.74)–(6.91)] and Inexact-Equations—[Compare (6.92)–(6.241)]—$Riccati^{4.}$ equations—[Compare (6.242)–(6.268)]—$Euler^{7.}$ equations—[Compare (6.269)–(6.315)] , and the Factorable Equations—[Compare (6.316)–(6.344)].

Described below are a few equations that represent special situations. Such situations are slightly different from the special types treated in the preceding chapter. And with these situations in place, satisfactory solution of relevant differential equations is often possible.

Simple Cases—see (7.1)–(7.27)—are studied first. These cases are either readily integrated, or they have the independent and/or the dependent variables missing in explicit form.

In (7.29)–(7.47), one works with an interesting new situation that is referred to as order reduction. Generally, it is not very likely that one of the non-trivial solution to a given differential equation is known. However, if it should happen, the given equation can be reduced to one which is of order one lower than the original. This is particularly helpful when dealing with a first-degree, second-order homogeneous linear (ODE). Because with prior knowledge of one non-trivial solution, the equation evolves into one of the first order. And such first-order differential equations are often more manageable than the original second-order equations.

Should a differential equation with variable coefficients also be inhomogeneous, finding its complete solution would require knowledge of both its complementary solution and the particular integral. While the method of undetermined coefficients—see (3.60)–(3.91)—worked well for determining the I_{pi} for equations with constant coefficients, differential equations with variable coefficients, in contrast, are best treated by a procedure termed variation of parameters . This latter procedure is described and extensively illustrated by relevant application—see (7.48) → (7.62).

Finally, two different varieties of second-order differential equations are considered where a known special situation obtains. For known special situations, the second-order differential equation can often be solved by following particular routines which are special to that case. These routines are described in detail in (7.63)–(7.124).

7.2 Simple Cases

Equation (1)

Consider

$$\left[D^2 y\right]^n \;=\; \Phi(x) \;, \tag{7.1}$$

or, equivalently, the second-order, first-degree ordinary differential equation

$$(1) \;:\qquad \left[D^2 y\right] \;=\; f(x) \;. \tag{7.2}$$

Here, as usual, $D = \frac{d}{\mathrm{d}x}$. Clearly, differential equations of the form (7.2) can directly be integrated. First and second integration yields

$$\int [D^2 y]\mathrm{d}x \;=\; Dy \;=\; \int f(x)\mathrm{d}x + \sigma_1 \;,$$
$$\int [Dy]\mathrm{d}x \;=\; y \;=\; \int \mathrm{d}x \left\{ \int f(x)\mathrm{d}x \right\} + \sigma_1 x + \sigma_2 \;. \tag{7.3}$$

As an exercise, in (7.2), try

$$f(x) = x_0 - x \;. \tag{7.4}$$

Then, (7.3) leads to the result

$$y = \frac{x_0\, x^2}{2} - \frac{x^3}{6} + \sigma_1 x + \sigma_2 \;. \tag{7.5}$$

Equation (2)

Next, treat equations of the form

$$(2) \;:\; D^2 y(x) = f\left(Dy(x)\right) \;. \tag{7.6}$$

Use the notation $D = \frac{d}{\mathrm{d}x}$, $q(x) = \frac{dy(x)}{\mathrm{d}x} = Dy(x)$, and rewrite (7.6) as

$$\frac{dq(x)}{\mathrm{d}x} = f[q(x)]$$

and integrate.

$$\int \frac{dq(x)}{f\,[q(x)]} = x \;+\; \text{constant} \;. \tag{7.7}$$

The left-hand side of (7.7) is some function $F[q(x)]$ that in principle determines $q(x)$. Once $q(x)$, $= \frac{dy(x)}{\mathrm{d}x}$, is known, its integration would lead to the desired solution $y(x)$.

Equation (3)

$$\left[D^2 y\right]^2 = 1 + (Dy)^2 \quad . \tag{7.8}$$

A more suitable form, convenient for calculation, is the square root of (7.8).

$$\begin{aligned} (3) \ : \qquad D^2 y(x) = Dq(x) \ &= \ \frac{d\, q(x)}{\mathrm{d}\, x} \\ &= \pm\sqrt{1 + [q(x)]^2} \ . \end{aligned} \tag{7.9}$$

Or, equivalently,

$$\frac{dq(x)}{\sqrt{1 + [q(x)]^2}} = \pm\, \mathrm{d}x \ . \tag{7.10}$$

Integration leads to

$$\sinh^{-1}[q(x)] = \pm x + \text{constant} \ . \tag{7.11}$$

Inverting (7.11) yields

$$q(x) \ = \ \frac{dy(x)}{\mathrm{d}x} = \sinh\left[x + \sigma_1\right] \ . \tag{7.12}$$

And final integration with respect to x leads to

$$y(x) = \cosh\left(x + \sigma_1\right) + \sigma_2 \ . \tag{7.13}$$

7.2.1 Problems Group I

Solve the following (ODE).

$$\left[D^2 y\right]^2 = 1 + (Dy) \quad .$$

7.2.2 Equations (a) and (b)

Consider second-order, first-degree linear (ODE) of the form

$$D^2 y(x) + \Phi(x)\, Dy(x) = 0 \ .$$

This equation is best re-expressed as a first-order, first-degree linear (ODE) in terms of $q(x)$—meaning, $\frac{dy(x)}{\mathrm{d}x}$—as the new dependent variable.

$$Dq(x) + \Phi(x)\, q(x) = 0 \ , \tag{7.14}$$

Once solved, $q(x)$ can be integrated with respect to x to find $y(x)$. Below, we solve the following first-degree linear (ODE)'s.

$$\begin{aligned} (a): \quad & D^2y + (x + x^2)\, Dy \ = \ 0 \ ; \\ (b): \quad & D^2y + \exp(x)\, Dy \ = \ 0 \ . \end{aligned} \tag{7.15}$$

In the following, we define $M(x)$, $N(x)$, and $W(x)$ and work out the solution.

7.2.3 Solution

$$\begin{aligned} (a) \ : \quad & Dq(x) + (x + x^2)\, q(x) \ = \ 0 \ . \\ M(x) \equiv\ & x + x^2 \ ; \quad N(x) \ = \ 0 \ ; \\ W(x) \equiv\ & \exp\left(\int M(x)\, \mathrm{d}x \right) \ = \ \exp\left(\frac{x^2}{2} + \frac{x^3}{3} \right) \ ; \\ q(x) \equiv\ & \frac{1}{W(x)} \left[\int W(x)\, N(x)\, \mathrm{d}x + \sigma_1 \right] \\ =\ & \sigma_1 \, \exp\left(-\frac{x^2}{2} - \frac{x^3}{3} \right) \ . \end{aligned}$$

Integrating $q(x)$ with respect to x leads to the desired solution.

$$y(x) \ = \ \int q(x)\, \mathrm{d}x = \sigma_1 \int \exp\left(-\frac{x^2}{2} - \frac{x^3}{3} \right) \mathrm{d}x + \sigma_2 \ . \tag{7.16}$$

$$\begin{aligned} (b) \ : \quad & Dq(x) + \exp(x)\, q(x) \ = \ 0 \ . \\ M(x) \equiv\ & \exp(x) \ ; \quad N(x) \ = \ 0 \ ; \\ W(x) \equiv\ & \exp\left(\int M(x)\, \mathrm{d}x \right) \ = \ \exp\{\exp(x)\} \ ; \\ q(x) \equiv\ & \frac{1}{W(x)} \left[\int W(x)\, N(x)\, \mathrm{d}x + \sigma_1 \right] \\ =\ & \sigma_1 \, \exp\{-\exp(x)\} \ . \end{aligned}$$

Therefore, the solution $y(x)$ is

$$y(x) \;=\; \int q(x)\mathrm{d}x \;=\; \sigma_1 \int \exp\{-\exp(x)\}\mathrm{d}x + \sigma_2 \;. \tag{7.17}$$

Clearly, this procedure would work just as well for other choices of the function $\Phi(x)$. To that end, it is helpful to set $z = -\exp(x)$. Then, represent $\int \exp\{-\exp(x)\}\mathrm{d}x$ as $\int \frac{\exp(z)}{z}\mathrm{d}z = \sigma_3\ \mathrm{Ei}[z] = \sigma_3\ \mathrm{Ei}[\text{-}\exp(x)]$.

7.2.4 *Equations (c) and (d)*

Solve the following second-order, first-degree linear (ODE).

$$\begin{aligned} (c): &\qquad D^2y(x) + \frac{1}{y(x)}\,Dy(x) \;=\; 0 \;. \\ (d): &\qquad D^2y(x) + 2\,y(x)\,.\,Dy(x) \;=\; 0 \;. \end{aligned} \tag{7.18}$$

Solution (c)
Because

$$D^2y(x) \;=\; Dq(x) \;=\; \frac{dq(x)}{\mathrm{d}x} \;=\; \frac{dq(x)}{dy(x)}\,.\,\frac{dy(x)}{\mathrm{d}x} \;=\; \frac{dq(x)}{dy(x)}\,.\,q(x) \;, \tag{7.19}$$

and

$$\begin{aligned} (c): &\quad \frac{dq(x)}{dy(x)}\,.\,q(x) + \frac{1}{y(x)}\,.\,Dy(x) \\ &= \frac{dq(x)}{dy(x)}\,.\,q(x) + \frac{1}{y(x)}\,.\,q(x) \;=\; 0 \;. \end{aligned} \tag{7.20}$$

Therefore,

$$(c): \qquad \frac{dq(x)}{dy(x)} = -\frac{1}{y(x)} \;. \tag{7.21}$$

Integrate the above with respect to $y(x)$.

$$\int \frac{dq(x)}{dy(x)}\,.\,dy(x) \;=\; \int dq(x) = -\int \frac{dy(x)}{y(x)} \;.$$

We get

$$q(x) = -\log[y(x)] - \sigma_0 \;.$$

Invert the above

$$\frac{1}{q(x)} \equiv \frac{\mathrm{d}x}{dy(x)} = -\frac{1}{\log[y(x)] + \sigma_0} \ ,$$

multiply both sides by $dy(x)$ and integrate

$$\int \mathrm{d}x = -\int \frac{dy(x)}{\log[y(x)] + \sigma_0} \ .$$

[Note it is helpful to know also the equality:

$$-\int \frac{dy(x)}{\log[y(x)]+\sigma_0} \ = \ -\exp(-\sigma_0) \int \frac{\exp(\log[y(x)]+\sigma_0)}{\log[y(x)]+\sigma_0} \, d(\log[y(x)] + \sigma_0).]$$

The result is

$$(c) \ : \quad x + \sigma_1 \ = \ -\exp(-\sigma_0) \, Ei \left[\log y(x) + \sigma_0\right] \ . \tag{7.22}$$

Solution (d)

$$(d) : \qquad D^2 y(x) + 2\, y(x) \, . \, Dy(x) = 0 \qquad . \tag{7.23}$$

Equation (d) can be shown to lead to

$$\frac{dq(x)}{dy(x)} = -2\, y(x) \ . \tag{7.24}$$

Integration with respect to y gives

$$\begin{aligned} \int \frac{dq(x)}{dy(x)} \, . \, dy(x) \ &= \ q(x) \ = \ \left[\frac{dy(x)}{\mathrm{d}x}\right] \\ = \ -2 \int y(x) \, . \, dy(x) &= \ (\sigma_0)^2 - y(x)^2 \end{aligned} \tag{7.25}$$

where $(\sigma_0)^2$ is a constant. Invert (7.25) and integrate with respect to $y(x)$.

$$\begin{aligned} \int \left[\frac{\mathrm{d}x}{dy(x)}\right] . \, dy(x) = \int \mathrm{d}x \ &= \ \int \frac{dy(x)}{(\sigma_0)^2 - y(x)^2} \ , \\ = \ x \ &= \left(\frac{1}{\sigma_0}\right) \tanh^{-1}\left[\frac{y(x)}{\sigma_0}\right] - \sigma_1 \ . \end{aligned} \tag{7.26}$$

Thus, the final result

$$(d) \ : \qquad y(x) = \sigma_0 \ \tanh\left[\sigma_0 \cdot (x + \sigma_1)\right] \ . \tag{7.27}$$

7.2.5 *Problems Group II*

Solve the following two differential equations (1) and (2).

$$\begin{aligned} (1): \quad & D^2y + y\,Dy = 0 \ ; \\ (2): \quad & D^2y + \exp(y)\,Dy = 0 \ . \end{aligned} \tag{7.28}$$

7.3 Order Reduction

Occasionally, the unlikely is true and one of the ν non-zero solutions to a νth order homogeneous linear (ODE) is already known. When that happens, the given equation can be reduced to an equation of the $(\nu - 1)$th order.

Consider, for example, a νth order homogeneous linear (ODE)

$$\sum_{i=0}^{\nu} a_i(x) D^i u(x) = 0 \tag{7.29}$$

where $\{a_i(x)\}$, $i = 0, 1, ..., \nu$ are all known functions of x. The solution $u(x)$ of (7.29) is of course unknown. Given $g(x)$ is a known solution of (7.29). Consider a function $\phi(x)$ that is a product of $g(x)$ and a new function $f(x)$.

$$\phi(x) = f(x)\, g(x) \ . \tag{7.30}$$

Assume $f(x)$ is so chosen that $\phi(x)$ is also a solution of (7.29). This requirement results in $Df(x)$ satisfying homogeneous linear (ODE) of the $(\nu - 1)$th order. Because the size of the relevant algebra decreases rapidly with decrease in order, for simplicity one works with a low value of ν. Accordingly, in the following, only a second-order linear (ODE)is treated.

$$\left[a_0(x) + a_1(x)\, D + a_2(x)\, D^2\right] u(x) \ = \ 0 \ . \tag{7.31}$$

The two solutions to this equation are notated $u_1(x)$ and $u_2(x)$. Regarding these solutions:
(a): Assume $u_1(x)$ is a known (or correctly guessed) non-trivial solution.

$$\left[a_0(x) + a_1(x)\, D + a_2(x)\, D^2\right] u_1(x) \ = \ 0 \tag{7.32}$$

(b): The other linearly independent solution, $u_2(x)$, to the same differential equation, that is

$$\left[a_0(x) + a_1(x)\, D + a_2(x)\, D^2\right] u_2(x) \ = \ 0 \ , \tag{7.33}$$

is currently unknown.

7.3.1 Linear Independence of u_1 and u_2

It was asserted above that functions $u_1(x)$ and $u_2(x)$—see (7.32) and (7.33)—are linearly independent. It is helpful to show that this assertion is correct.

According to (3.47), two functions $u_1(x)$ and $u_2(x) = f(x)\,u_1(x)$ are linearly independent if and only if their Wronskian is non vanishing. In other words, functions $u_1(x)$ and $u_2(x)$ are linearly independent only if the following is true

$$\begin{vmatrix} u_1(x) & u_2(x) \\ Du_1(x) & Du_2(x) \end{vmatrix} = \begin{vmatrix} u_1(x) & f(x)\,u_1(x) \\ Du_1(x) & D[f(x)\,u_1(x)] \end{vmatrix} = u_1^2(x) \cdot Df(x) \neq 0 \,. \tag{7.34}$$

With $f(x)$, as in (7.45), the right-hand side of (7.34) is

$$(1): \quad u_1^2(x) \cdot Df(x) = \sigma_0 \cdot \exp\left[-\int \frac{a_1(x)}{a_2(x)} \mathrm{d}x \right] . \tag{7.35}$$

As such, the Wronskian is non-zero as long as $a_1(x)$ is non-infinite and $a_2(x)$ is non-zero. Assuming both these requirements are satisfied, $u_1(x)$ and $u_2(x)$ are indeed linearly independent.

7.4 Reduce Order from Second to First

Represent the second solution as a product of the first and a new function $f(x)$. That is

$$u_2(x) = f(x)\,u_1(x) \quad . \tag{7.36}$$

Because $u_1(x)$ is already known, in order to determine $u_2(x)$ all one needs is calculate $f(x)$. And to that purpose, proceed as follows:

(A): Differentiate (7.36) twice.

$$\begin{aligned} Du_2(x) &= f(x)\,Du_1(x) + u_1(x)\,Df(x) \quad ; \\ D^2u_2(x) &= f(x)\,D^2u_1(x) + 2\,Du_1(x)\,Df(x) + u_1(x)\,D^2f(x) \,. \end{aligned} \tag{7.37}$$

(B): Substitute these differentials into (7.33).

$$\begin{aligned} 0 &= a_0\,f(x)\,u_1(x) + a_1(x)[f(x)\,Du_1(x) + u_1(x)\,Df(x)] \\ &+ a_2(x)\,[f(x)\,D^2u_1(x) + 2\,Du_1(x)\,Df(x) + u_1(x)\,D^2f(x)] \quad . \end{aligned} \tag{7.38}$$

(C): Rewrite (7.38). as

$$\begin{aligned} 0 = f(x)\,\left[a_0 + a_1(x)\,D + a_2(x)\,D^2\right]u_1(x) &+ a_1(x)\,u_1(x)\,Df(x) \\ &+ 2\,a_2(x)\,Du_1(x)\,Df(x) + a_2(x)\,u_1(x)\,D^2f(x) \,. \end{aligned} \tag{7.39}$$

As per (7.32), the first term on the right-hand side of (7.39) is equal to zero. By introducing a convenient notation,

$$v(x) = Df(x) \tag{7.40}$$

one can write the remaining parts of (7.39) as

$$[a_1(x)\,u_1(x) \;+\; 2\,a_2(x)Du_1(x)]\,v(x) \;+\; a_2(x)\,u_1(x)\,Dv(x) \;=0. \tag{7.41}$$

Dividing both sides by $a_2(x)u_1(x)v(x)$ and then multiplying by $\mathrm{d}x$ yields a neater version of (7.41). That is

$$\frac{a_1(x)}{a_2(x)}\mathrm{d}x + 2\,\frac{du_1(x)}{u_1(x)} + \frac{dv(x)}{v(x)} = 0\,. \tag{7.42}$$

Integration gives

$$\int \frac{a_1(x)}{a_2(x)}\mathrm{d}x + 2\,\log[u_1(x)] + \log[v(x)] = \log(\sigma_0)$$

and exponentiating both sides leads to

$$\exp\left[\,\int \frac{a_1(x)}{a_2(x)}\mathrm{d}x\right]\cdot\,[u_1(x)]^2\cdot v(x) \;=\sigma_0\;. \tag{7.43}$$

As such

$$v(x) \;=\; \sigma_0\,\frac{\exp\left[-\int \frac{a_1(x)}{a_2(x)}\mathrm{d}x\right]}{[u_1(x)]^2}\;. \tag{7.44}$$

Because according to (7.40), $v(x) = \frac{df(x)}{\mathrm{d}x}$, and therefore an integration with respect to x would give

$$\int v(x)\mathrm{d}x = \sigma_0\int \frac{\exp\left[-\int \frac{a_1(x)}{a_2(x)}\mathrm{d}x\right]}{[u_1(x)]^2}\,\mathrm{d}x = \int \frac{df(x)}{\mathrm{d}x}\,\mathrm{d}x = f(x)\;.$$

The conclusion: if one solution is known—that is, if $u_1(x)$ is already known—for a second-order linear ordinary differential equation, one can always find its second solution $u_2(x)$.

$$(D)\;:\;u_2(x) = \;u_1(x)\,f(x) \;=\; \sigma_0\,u_1(x)\cdot\int_0^x \frac{\exp\left[-\int \frac{a_1(x)}{a_2(x)}\mathrm{d}x\right]}{[u_1(x)]^2}\,\mathrm{d}x\;. \tag{7.45}$$

7.4.1 Examples (I)–(VII)

The seven differential equations and their first solution, $u_1(x)$, are given below. By using the procedure 'reduce order from second to first' work out the second solution $u_2(x)$.

$$\begin{aligned}
(I): &\quad (D^2+4)u(x) = 0\,. \;:\; u_1(x) = \sigma_1 \sin(2x)\\
(II): &\quad (D^2+b\,D-2\,b^2)u(x) = 0\,. \;:\; u_1(x) = \sigma_1 \exp(b\,x)\\
(III): &\quad [D^2+D-2]u(x) = 0\,. \;:\; u_1(x) = \sigma_1 \exp(x)\\
(IV): &\quad [D^2-2\,D+1]u(x) = 0\,. \;:\; u_1(x) = \sigma_1 \exp(x)\\
(V): &\quad (x^2\,D^2-x\,D-3)u(x) = 0\,. \;:\; u_1(x) = \sigma_1\, x^3\\
(VI): &\quad (D^2+x\,D+1)u(x) = 0\,. \;:\; u_1(x) = \sigma_1 \exp\left[-\frac{x^2}{2}\right]\\
(VII): &\quad (D^2+x\,D+x)u(x) = 0\,. \;:\; u_1(x) = \sigma_1 \exp\left[x-\frac{x^2}{2}\right] \qquad (7.46)
\end{aligned}$$

7.4.2 Solution

$$\begin{aligned}
(I): &\quad u_1(x) = \sigma_1 \sin(2x)\,;\; a_2(x)=1\,;\; a_1(x)=0\,,\; a_0(x)=4\;;\\
&\quad u_2(x) = \sigma_1 \sin(2x)\cdot\sigma_0 \int \frac{\exp\left[-\int\left(\frac{0}{1}\right)\mathrm{d}x\right]}{[\sigma_1 \sin(2x)]^2}\,\mathrm{d}x\\
= &\quad \sin(2x)\cdot\int \frac{\sigma_3}{[\sin(2x)]^2}\,\mathrm{d}x = \sigma_2\cos(2x) = u_2(x) \quad :(1)\;.
\end{aligned}$$

$$\begin{aligned}
(II): &\quad u_1(x) = \sigma_1 \exp(b\,x)\,;\; a_2(x)=1\,;\; a_1(x)=b\,,\; a_0(x)=-2b^2\;;\\
&\quad u_2(x) = \sigma_1 \exp(b\,x)\cdot\sigma_0 \int \frac{\exp\left[-\int\left(\frac{b}{1}\right)\mathrm{d}x\right]}{[\sigma_1 \exp(b\,x)]^2}\,\mathrm{d}x\\
&\qquad = \sigma_2 \exp(-2\,b\,x) = u_2(x)\;. \qquad :(2)\;.
\end{aligned}$$

$$\begin{aligned}
(III): &\quad u_1(x) = \sigma_1 \exp(x)\,;\; a_2(x)=1\,;\; a_1(x)=1\,,\; a_0(x)=-2\;;\\
&\quad u_2(x) = \sigma_1 \exp(x)\cdot\sigma_0 \int \frac{\exp\left[\int -\left(\frac{1}{1}\right)\mathrm{d}x\right]}{[\exp(x)]^2}\,\mathrm{d}x\\
&\qquad = \sigma_2 \exp(-2\,x) = u_2(x) \qquad :(3)\;.
\end{aligned}$$

$$
\begin{aligned}
(IV): \quad u_1(x) &= \sigma_1 \exp(x)\ ;\ a_2(x) = 1\ ;\ a_1(x) = -2\ ,\ a_0(x) = 1\ ; \\
u_2(x) &= \sigma_1 \exp(x) \cdot \sigma_0 \int \frac{\exp\left[\int (2)\mathrm{d}x\right]}{[\exp(x)]^2}\,\mathrm{d}x \\
&= \sigma_2\, x \exp(x) = u_2(x) \qquad\qquad : (4)\ .
\end{aligned}
$$

$$
\begin{aligned}
(V): \ u_1(x) &= \sigma_1\, x^3\ ;\ a_2(x) = x^2\ ;\ a_1(x) = -x\ ,\ a_0(x) = -3\ ; \\
u_2(x) &= \sigma_1\, x^3 \cdot \sigma_0 \int \frac{\exp\left[-\int \left(\frac{-x}{x^2}\right)\mathrm{d}x\right]}{[\sigma_1\, x^3]^2}\,\mathrm{d}x = \frac{\sigma_2}{x} \qquad : (5)\ .
\end{aligned}
$$

$$
\begin{aligned}
(VI): \quad u_1(x) &= \sigma_1 \exp\left(-\frac{x^2}{2}\right)\ ;\ a_2(x) = 1\ ;\ a_1(x) = x\ ,\ a_0(x) = 1\ ; \\
u_2(x) &= \sigma_1 \exp\left(-\frac{x^2}{2}\right) \cdot \sigma_0 \int \frac{\exp\left[\int -\left(\frac{x}{1}\right)\mathrm{d}x\right]}{[\exp(-x^2)]}\,\mathrm{d}x \\
&= \sigma_2 \exp\left(-\frac{x^2}{2}\right) \cdot \int \exp\left(\frac{x^2}{2}\right)\mathrm{d}x = u_2(x) \qquad : (6)\ .
\end{aligned}
$$

$$
\begin{aligned}
(VII): \quad u_1(x) &= \sigma_1 \exp\left(x - \frac{x^2}{2}\right)\ ;\ a_2(x) = 1\ ;\ a_1(x) = x\ ,\ a_0(x) = x\ ; \\
u_2(x) &= \sigma_1 \exp\left(x - \frac{x^2}{2}\right) \cdot \sigma_0 \int \frac{\exp\left[\int (-x)\mathrm{d}x\right]}{[\exp\left(x - \frac{x^2}{2}\right)]^2}\,\mathrm{d}x \\
&= \sigma_2 \exp\left(x - \frac{x^2}{2}\right) \cdot \int \exp\left[\frac{x^2}{2} - 2x\right]\mathrm{d}x = u_2(x) \qquad : (7)\ .
\end{aligned}
\tag{7.47}
$$

7.4.3 Problems Group III

Given one solution, $u_1(x)$, find the second solution, $u_2(x)$, of the following differential equations. [Hint: Read (7.32)–(7.47).]

(1) : $(D^2 + 1)u(x) = 0\ .\ :\ u_1(x) = \sigma_1 \sin(x)$

(2) : $(D^2 + 2\,D - 1)u(x) = 0\ .\ :\ u_1(x) = \sigma_1 \exp\left\{\left(\sqrt{2} - 1\right)x\right\}$

(3) : $\left(D^2 + D + \frac{1}{2}\right)u(x) = 0\ .\ :\ u_1(x) = \sigma_1 \exp\left(-\frac{x}{2}\right)\sin\left(\frac{x}{2}\right)$

(4) : $(D^2 - 2\,D + 2)u(x) = 0\ .\ :\ u_1(x) = \sigma_1 \exp(x)\sin(x)$

(5) : $(x^2\,D^2 - x\,D - 1)u(x) = 0\ .\ :\ u_1(x) = \sigma_1\, x^{1+\sqrt{2}}$

$$(6) \quad : \quad (D^2 - x\,D - 1)u(x) \;=\; 0\,. \quad : \quad u_1(x) \;=\; \sigma_1 \exp\left(\frac{x^2}{2}\right)$$

$$(7) \quad : \quad (D^2 + x\,D - x)u(x) \;=\; 0\,. \quad : \quad u_1(x) \;=\; \sigma_1 \exp\left(-x - \frac{x^2}{2}\right).$$

7.4.4 Particular Integral

Second-order homogeneous linear ordinary differential equations were treated in the preceding section. Similar homogeneous linear (ODE)'s that are made inhomogeneous are treated here. The latter is done by the addition of a known function, $B(x)$, to the right-hand side in place of the zero that normally appears in homogeneous linear (ODE)'s.

If $S_{cs}(x)$ is a complete solution of such equation—compare, (2.5)—one would have

$$\left[a_0 + a_1(x)\,D + a_2(x)\,D^2\right] S_{cs}(x) \;=\; B(x) \quad . \tag{7.48}$$

As noted in (3.59)—also compare (2.16)—the complete solution, $S_{cs}(x)$ is the sum of the complementary solution, $S_{\text{comp}}(x) \equiv u_1(x) + u_2(x)$, and a particular integral, $I_{pi}(x)$. That is

$$S_{cs}(x) \;=\; S_{\text{comp}}(x) + \; I_{pi}(x) \;=\; u_1(x) + u_2(x) \;+\; I_{pi}(x) \quad .$$

Thus, in addition to solving the differential equations

$$\begin{aligned} \left[a_0 + a_1(x)\,D + a_2(x)\,D^2\right] u_1(x) \;&=\; 0 \quad , \\ \left[a_0 + a_1(x)\,D + a_2(x)\,D^2\right] u_2(x) \;&=\; 0 \quad , \end{aligned} \tag{7.49}$$

one would also need to solve

$$\left[a_0 + a_1(x)\,D + a_2(x)\,D^2\right] I_{pi}(x) \;=\; B(x) \quad . \tag{7.50}$$

7.4.5 Calculation of $I_{pi}(x)$ Variation of Parameters

As per our previous experience determining $I_{pi}(x)$ for inhomogeneous linear (ODE)'s with constant coefficients—see (3.60)–(3.91)—it is reasonable to expect the current particular integral, $I_{pi}(x)$, will also bear relationship to the solution of the homogeneous linear (ODE) (7.49): namely $u_1(x)$ and $u_2(x)$. With this expectation in mind, introduce a trial solution of the form

$$I_{pi}(x) = f_1(x)\,u_1(x) + f_2(x)\,u_2(x) \quad , \tag{7.51}$$

where $f_1(x)$ and $f_2(x)$ are unknown functions that need to be determined. Clearly, in order to determine $f_1(x)$ and $f_2(x)$, two independent differential equations that involve these functions are required.

Another important point to note is that the basic differential equation that specifies $I_{pi}(x)$—that is (7.50)—is second order. Therefore, one would need to differentiate the trial solution (7.51) twice.

7.4.6 Procedure

Differentiate (7.51).

$$\begin{aligned} DI_{pi}(x) &= f_1(x)\,Du_1(x) + f_2(x)\,Du_2(x) \\ &\quad + [u_1(x)\,Df_1(x) + u_2(x)\,Df_2(x)] \quad . \end{aligned} \tag{7.52}$$

Differentiating once again would obviously introduce $D^2 f_1(x)$ and $D^2 f_2(x)$. But most certainly, unless it cannot be avoided, nobody wants to be stuck with having to deal with second-order differential equations for the yet to be determined functions $f_1(x)$ and $f_2(x)$. Therefore, if at all possible, second differentials of $f_1(x)$ and $f_2(x)$ must be avoided. Obviously, therefore, one must set that part of (7.52) equal to zero which upon differentiation would introduce the unwanted second differentials. This implies setting

$$\{FIRST\} \;:\; [u_1(x)\,Df_1(x) + u_2(x)\,Df_2(x)] = 0 \;. \tag{7.53}$$

As a result, one now has

$$\begin{aligned} DI_{pi}(x) &= f_1(x)\,Du_1(x) + f_2(x)\,Du_2(x) \quad ; \\ D^2 I_{pi}(x) &= Df_1(x)\,Du_1(x) + Df_2(x)\,Du_2(x) \\ &\quad + f_1(x)\,D^2 u_1(x) + f_2(x)\,D^2 u_2(x) \quad . \end{aligned} \tag{7.54}$$

Rewrite the original differential equation, namely (7.50), by making use of (7.51) and (7.54). That is, write

$$\begin{aligned} B(x) &= [a_0(x) + a_1(x)D + a_2(x)D^2]I_{pi}(x) \qquad ; \\ &= a_0(x)[f_1(x)\,u_1(x) + f_2(x)\,u_2(x)] \\ &\quad + a_1(x)[f_1(x)\,Du_1(x) + f_2(x)\,Du_2(x)] \\ &\quad + a_2(x)\,[Df_1(x)\,Du_1(x) + Df_2(x)\,Du_2(x) + f_1(x)\,D^2 u_1(x) + f_2(x)\,D^2 u_2(x)] \end{aligned} \tag{7.55}$$

as

$$\begin{aligned} B(x) &= f_1(x)[a_0(x) + a_1(x)D + a_2(x)D^2]u_1(x) \\ &+ f_2(x)[a_0(x) + a_1(x)D + a_2(x)D^2]u_2(x) \\ &+ a_2(x)[Df_1(x)\,Du_1(x) + Df_2(x)\,Du_2(x)] \quad . \end{aligned} \tag{7.56}$$

Recall that each of the two functions $u_1(x)$ and $u_2(x)$ are solutions of the differential equation (7.49). As such, the top two lines on the right-hand side of (7.56) are zero, and the third can be written as

$$\{SECOND\} \;:\; Df_1(x)\,Du_1(x) + Df_2(x)\,Du_2(x) \;=\; \{B(x)/a_2(x)\} \quad . \tag{7.57}$$

Clearly, (7.57) is the desired $\{SECOND\}$ of the two differential equations—the first being (7.53) named $\{FIRST\}$—that $Df_1(x)$ and $Df_2(x)$ must satisfy.

Equations $\{FIRST\}$ and $(\{SECOND\}$ are simultaneous linear equations in two unknowns $Df_1(x)$ and $Df_2(x)$. Using elementary algebra, one finds

$$\begin{aligned} Df_1(x) = \frac{df_1}{dx} &= -\left[\frac{u_2(x)}{u_1(x)Du_2(x) - u_2(x)Du_1(x)}\right]\{B(x)/a_2(x)\}\,; \\ Df_2(x) = \frac{df_2}{dx} &= \left[\frac{u_1(x)}{u_1(x)Du_2(x) - u_2(x)Du_1(x)}\right]\{B(x)/a_2(x)\}\,. \end{aligned} \tag{7.58}$$

Integration leads to the functions $f_1(x)$ and $f_2(x)$. Recall that one started off already knowing the solution to the homogeneous part of the differential equation—namely $u_1(x)$ and $u_2(x)$—therefore the particular integral,

$$I_{pi}(x) \;=\; f_1(x)\,u_1(x) \;+\; f_2(x)\,u_2(x),$$

can now be evaluated.

7.4.7 *Examples:* $I_{pi}(x) - (1) \rightarrow I_{pi}(x) - (4)$

We determine the $I_{pi}(x)$ for examples numbered (1) - (4). Given alongside, these equations are the relevant $u_1(x)$ and $u_2(x)$: that is, both the first and the second solution to the homogeneous part of each of these differential equations. Note that these equations have constant coefficients. Therefore, they can be solved by using 'undetermined coefficients' that were described in detail in (3.60)–(3.91). Indeed, for inhomogeneous linear (ODE)'s with constant coefficients, undetermined coefficients are much easier to use than the 'variation of parameters' procedure currently under discussion. Still, for demonstrational purposes, we use the more laborious variation of parameters procedure described above.

$$
\begin{aligned}
I_{pi}(x) - (1) \quad &: \quad (D^2+4)u(x) \;=\; x\exp(x)\;; \\
&\quad\; u_1(x) \;=\; \sigma_1\,\sin(2x)\;;\; u_2(x) \;=\; \sigma_2\,\cos(2x) \quad .(1) \\
I_{pi}(x) - (2) \quad &: \quad (D^2+b\,D-2\,b^2)u(x) \;=\; \exp(x)\;; \\
&\quad\; u_1(x) \;=\; \sigma_1\,\exp(b\,x)\;;\; u_2(x) \;=\; \sigma_2\,\exp(-2bx).\,(2) \\
I_{pi}(x) - (3) \quad &: \quad (D^2+D-2)u(x) \;=\; 1/x\;; \\
&\quad\; u_1(x) \;=\; \sigma_1\,\exp(x)\;;\; u_2(x) \;=\; \sigma_2\,\exp(-2x) \quad .(3) \\
I_{pi}(x) - (4) \quad &: \quad (D^2-2\,D+1)u(x) \;=\; x\;; \\
&\quad\; u_1(x) \;=\; \sigma_1\,\exp(x)\;;\; u_2(x) \;=\; \sigma_2\,x\,\exp(x) \quad .(4)
\end{aligned}
\tag{7.59}
$$

7.4.8 Solution

Employe (7.58), calculate $Df_1(x)$ and $Df_2(x)$, integrate the result, and determine $f_1(x)$ and $f_2(x)$.

$$
\begin{aligned}
I_{pi}(x) - (1) \quad &: \quad Df_1(x) \;=\; \left(\frac{x}{2\sigma_1}\right)\exp(x)\,\cos(x)\;, \\
Df_2(x) \;&=\; -\left(\frac{x}{2\sigma_2}\right)\exp(x)\,\sin(x)\;, \\
f_1(x) \;&=\; \left(\frac{\exp(x)}{50\sigma_1}\right)[(10x-4)\sin(2x)+(5x+3)\cos(2x)]+\sigma_3\;, \\
f_2(x) \;&=\; -\left(\frac{\exp(x)}{50\sigma_2}\right)[(5x+3)\sin(2x)+(4-10x)\cos(2x)]+\sigma_4\;.
\end{aligned}
$$

Given $u_1(x) \;=\; \sigma_1\,\sin(2x)$ and $u_2(x) \;=\; \sigma_2\,\cos(2x)$—see equation $I_{pi}(x) - (1)$ in Examples $I_{pi}(x)$ —the relevant $I_{pi}(x)$ is readily found.

$$
\begin{aligned}
I_{pi}(x) - (1) \quad &: \quad I_{pi}(x) \;=\; f_1(x)\,u_1(x)+f_2(x)\,u_2(x) \\
&\qquad =\; \frac{x\exp(x)}{5}-\frac{2\exp(x)}{25}+\sigma_3\,u_1(x)+\sigma_4\,u_2(x)\;.
\end{aligned}
$$

$$
\begin{aligned}
I_{pi}(x) - (2) \quad &: \quad Df_1(x) \;=\; \left(\frac{1}{3\,b\,\sigma_1}\right)\exp[(1-b)x]\;, \\
Df_2(x) \;&=\; -\left(\frac{1}{3\,b\,\sigma_2}\right)\exp[(1+2\,b)x]\,, \\
f_1(x) \;&=\; \left(\frac{1}{3\,b\,\sigma_1}\right)\left(\frac{1}{1-b}\right)\exp[(1-b)x] \;+\; \sigma_3 \\
f_2(x) \;&=\; -\left(\frac{1}{3\,b\,\sigma_2}\right)\left(\frac{1}{1+2\,b}\right)\exp[(1+2\,b)x] \;+\; \sigma_4\;.
\end{aligned}
$$

Given $u_1(x) = \sigma_1 \exp(b\,x)$ and $u_2(x) = \sigma_2 \exp(-2\,b\,x)$—see equation $I_{pi}(x) - (2)$ in examples $I_{pi}(x)$ —the relevant $I_{pi}(x)$ is

$$\begin{aligned} I_{pi}(x) - (2) \quad : \quad I_{pi}(x) &= f_1(x)\,u_1(x) + f_2(x)\,u_2(x) \\ &= \tfrac{\exp(x)}{3b}\left[\tfrac{1}{1-b} - \tfrac{1}{1+2b}\right] + \sigma_3\,u_1(x) + \sigma_4\,u_2(x)\,. \end{aligned}$$

$$\begin{aligned} I_{pi}(x) - (3) \quad : \quad Df_1(x) &= \left(\frac{1}{3\,\sigma_1}\right)\frac{\exp(x)}{x}\,, \\ Df_2(x) &= -\left(\frac{1}{3\,\sigma_2}\right)\frac{\exp(2\,x)}{x}\,, \\ f_1(x) &= \left(\frac{1}{3\,\sigma_1}\right) Ei(-x) + \sigma_3 \\ f_2(x) &= -\left(\frac{1}{3\,\sigma_2}\right) Ei(2\,x) + \sigma_4\,. \end{aligned}$$

Given $u_1(x) = \sigma_1 \exp(x)$ and $u_2(x) = \sigma_2 \exp(-2\,x)$,—see equation $I_{pi}(x) - (3)$ in examples $I_{pi}(x)$ — the relevant $I_{pi}(x)$ is

$$\begin{aligned} I_{pi}(x) \quad - \quad (3) : I_{pi}(x) &= f_1(x)\,u_1(x) + f_2(x)\,u_2(x) \\ &= \frac{\exp(x)}{3} Ei(-x) - \frac{1}{3}\exp(-2\,x)\,Ei(2\,x) + \sigma_3\,u_1(x) + \sigma_4\,u_2(x)\,. \end{aligned}$$

Given $u_1(x) = \sigma_1 \exp(x)$ and $u_2(x) = \sigma_2\,x \exp(x)$—see equation $I_{pi}(x) - (4)$ in examples $I_{pi}(x)$ —the relevant $I_{pi}(x)$ is found by following a similar procedure to that used above.

$$\begin{aligned} I_{pi}(x) - (4) \quad : \quad Df_1(x) &= -\left(\frac{1}{\sigma_1}\right) x^2 \exp(-x)\,, \\ Df_2(x) &= -\left(\frac{1}{\sigma_2}\right) x \exp(-x)\,, \\ f_1(x) &= \left(\frac{1}{\sigma_1}\right)(x^2 + 2x + 2)\exp(-x) + \sigma_3 \\ f_2(x) &= -\left(\frac{1}{\sigma_2}\right)(x+1)\exp(-x) + \sigma_4 \quad . \end{aligned}$$

Therefore, one has

$$\begin{aligned} I_{pi}(x) - (4) : \; I_{pi}(x) &= f_1(x)\,u_1(x) + f_2(x)\,u_2(x) \\ &= (x^2 + 2x + 2) - x(x+1) + \sigma_3\,u_1(x) + \sigma_4\,u_2(x)\,. \\ &= x + 2 + \sigma_3\,u_1(x) + \sigma_4\,u_2(x) \end{aligned} \tag{7.60}$$

7.4.9 Examples: $I_{pi}(x) - (5) \rightarrow I_{pi}(x) - (10)$

Somewhat more complicated equations than those numbered (1)–(4) are treated below. Note: given alongside, these differential equations are $u_1(x)$ and $u_2(x)$, which are the two solutions to the homogeneous part of each of these differential equations.

(5) $(x^2\,D^2 - x\,D - 3)u(x) \;=\; \left(\dfrac{1}{x^2}\right)\;;\; u_1(x) = \sigma_1\,x^3\;;\; u_2(x) \;=\; \left(\dfrac{\sigma_2}{x}\right)$

(6) $(x^2\,D^2 + x\,D + 1)u(x) = \log(x)\;;$
$u_1(x) = \sigma_1\;\sin[\log(x)]\;;\; u_2(x) = \sigma_2\;\cos[\log(x)]$

(7) $[x^2 D^2 + x\,D + 4]u(x) = x^2\;\log(x)\;;$
$u_1(x) = \sigma_1\;\sin\left[2\;\log(x)\right]\;;\; u_2(x) = \sigma_2\;\cos\left[2\;\log(x)\right]$

(8) $[x(x+1)\,D^2 - (x+1)\,D + 1]u(x) \;=\; \left(\dfrac{1}{x}\right)\;;\; u_1(x) = \sigma_1\,(x+1)\;;$
$u_2(x) \;=\; \sigma_2[x\log(x+1) + \log(x+1) + 1]$

(9) $[(x^2-1)\,D^2 - (x^2 + 2x - 1)\,D + 2x]u(x) \;=\; x\;;\; u_1(x) = \sigma_1\,(x+1)^2\;;$
$u_2(x) \;=\; \sigma_2\;\exp(x) \qquad :$

(10) $[(x+2)^2\,D^2 + (x+2)\,D + 1]u(x) = (x+2)\;;$
$u_1(x) \;=\; \sigma_1\;\sin\left[\log(x+2)\right]\;;\; u_2(x) \;=\; \sigma_2\;\cos\left[\log(x+2)\right].$ (7.61)

These differential equations, namely (7.61)-(5)–(7.61)-(10), do not have constant coefficients. Therefore, for determining their $I_{pi}(x)$, it is appropriate to use the 'variation of parameters' procedure. Recall that the variation of parameters procedure was discussed in detail when equations (1)–(4) were solved above.

7.4.10 Solution

As before, use (7.58); calculate $Df_1(x)$ and $Df_2(x)$; integrate the result; and determine $f_1(x)$ and $f_2(x)$.

$$I_{pi}(x) \;-\; (5):\quad f_1(x) \;=\; -\left(20\sigma_1\,x^5\right)^{-1} \;+\; \sigma_3\;\;;\;\; f_2(x) \;=\; (4\sigma_2\,x)^{-1} \;+\sigma_4\,.$$

Using the information: $u_1(x) = \sigma_1\,x^3$, $u_2(x) \;=\; \frac{\sigma_2}{x}$ and

$$I_{pi}(x) \;-\; (5) = f_1(x)\,u_1(x) + f_2(x)\,u_2(x) \quad ,$$

one gets

$$I_{pi}(x) \;-\; (5):\; I_{pi}(x) = -\frac{1}{20\,x^2} + \frac{1}{4\,x^2} \;=\; \frac{x^{-2}}{5} + \;\sigma_3\,u_1(x) + \;\sigma_4\,u_2(x)\;\;.$$

Next, consider equation $I_{pi}(x) \; - \; (6)$.

$$I_{pi}(x) \; - \; (6): \quad f_1(x) = \frac{1}{\sigma_1} \{\log(x) \sin[\log(x)] + \cos[\log(x)]\} \; + \; \sigma_3 \; ;$$
$$f_2(x) = -\frac{1}{\sigma_2} \{\sin[\log(x] - \log(x) \cos[\log(x)]\} \; + \sigma_4 \; .$$

And the fact that $u_1(x) = \sigma_1 \; \sin[\log(x)]$, $u_2(x) = \sigma_2 \; \cos[\log(x)]$ and

$$I_{pi}(x) \; - \; (6) = f_1(x) \, u_1(x) + f_2(x) \, u_2(x)$$

leads to the result

$$I_{pi}(x) \; - \; (6): \quad I_{pi}(x) = \log(x) \; + \; \sigma_3 \, u_1(x) + \; \sigma_4 \, u_2(x) \quad .$$

In similar fashion equation $I_{pi}(x) \; - \; (7)$ leads to the following:

$$I_{pi}(x) \; - \; (7): \; f_1(x) = \frac{x^2}{16\sigma_1} \{[2\log(x) - 1] \sin[2\log(x)] + 2\log(x) \cos[2\log(x)]\} \; + \; \sigma_3 \; ;$$
$$f_2(x) = \frac{x^2}{16\sigma_2} \{[2\log(x) - 1] \cos[2\log(x)] - 2\log(x) \sin[2\log(x)]\} \; + \sigma_4 \; .$$

Therefore, using $u_1(x) = \sigma_1 \; \sin\left[2 \; \log(x)\right]$, $u_2(x) = \sigma_2 \; \cos\left[2 \; \log(x)\right]$ and the equation

$$I_{pi}(x) \; - \; (7) = f_1(x) \, u_1(x) + f_2(x) \, u_2(x) \; ,$$

one is led to

$$I_{pi}(x) \; - \; (7): \; I_{pi}(x) = \frac{x^2}{8} \log(x) - \frac{x^2}{16} \; + \; \sigma_3 \, u_1(x) + \; \sigma_4 \, u_2(x) \quad .$$

Proceeding as in equation $I_{pi}(x) \; - \; (6)$ one has

$$(8): \; f_1(x) = \frac{1}{\sigma_1} \left\{ \left(\frac{x^2 + 1}{2x^2} \right) \log(x + 1) - \frac{\log(x)}{2} + \frac{1 - x}{2x^2} \right\} \; + \; \sigma_3 \; ;$$
$$f_2(x) = \; - \; \frac{1}{\sigma_2} \left(\frac{1}{2x^2} \right) \; + \sigma_4 \; .$$

Therefore, using $u_1(x) = \sigma_1 \; \sin\left[2 \; \log(x)\right]$, $u_2(x) = \sigma_2 \; \cos\left[2 \; \log(x)\right]$ and the equation

$$I_{pi}(x) = f_1(x) \, u_1(x) + f_2(x) \, u_2(x)$$

gives

$$(7.61)-(8): \; I_{pi}(x) = \frac{1}{2}\left[(x+1)\log(x+1) - (x+1)\log(x) - 1\right] + \sigma_3\, u_1(x) + \sigma_4\, u_2(x)\,.$$

Proceeding as usual, (7.61)-(9) leads to

$$(9): \; f_1(x) = \left(\frac{1}{2\sigma_1}\right)\frac{1}{(x^2-1)} + \sigma_3 \;\; ;$$
$$f_2(x) = -\left(\frac{1}{\sigma_2}\right)\left[\frac{\exp(-x)}{x-1}\right] + \sigma_4 \quad .$$

Therefore, using $u_1(x) = \sigma_1\,(x+1)^2$, $u_2(x) = \sigma_2 \exp(x)$ and the relationship

$$I_{pi}(x) = f_1(x)\, u_1(x) + f_2(x)\, u_2(x)$$

one gets

$$(7.61)-(9): \qquad I_{pi}(x) = \frac{1}{2} + \sigma_3\, u_1(x) + \sigma_4\, u_2(x) \quad .$$

Finally, we treat (7.61)-(10).

$$(10): \; f_1(x) = \frac{1}{\sigma_1}\left(\frac{x+2}{2}\right)\{\sin[\log(x+2)] + \cos[\log(x+2)]\} + \sigma_3 \;\; ;$$
$$f_2(x) = -\frac{1}{\sigma_2}\left(\frac{x+2}{2}\right)\{\sin[\log(x+2)] - \cos[\log(x+2)]\} + \sigma_4 \quad .$$

Therefore, using $u_1(x) = \sigma_1 \sin[\log(x+2)]$, $u_2(x) = \sigma_2 \cos[\log(x+2)]$ and the relationship

$$I_{pi}(x) = f_1(x)\, u_1(x) + f_2(x)\, u_2(x)$$

one readily finds

$$(7.61)-(5): \qquad I_{pi}(x) = \frac{x}{2} + 1 + \sigma_3\, u_1(x) + \sigma_4\, u_2(x) \quad .$$

Note: Because $\sigma_3\, u_1(x) + \sigma_4\, u_2(x)$ are already present in the complementary solution, they can be ignored from the relevant result for $I_{pi}(x)$ for differential equations (7.59)-(1) $\rightarrow$ (7.59)-(10).

7.4.11 Problems Group IV

Determine the $I_{pi}(x)$ for the following inhomogeneous linear ordinary differential equation with constant coefficients numbered (1)–(10). Given alongside are $u_1(x)$ and $u_2(x)$: that is, both the first and the second solution to the homogeneous part of each of these differential equations. [Hint: Read (7.48)–(7.58).]

(1) $(D^2+1)u(x) = \exp(x)\,;\; u_1(x) = \sigma_1 \sin(x)\,;\; u_2(x) = \sigma_2 \cos(x)\,.$

(2) $(D^2+2\,D-2)u(x) = x^2\,;\; u_1(x) = \sigma_1 \exp\left\{\left(\sqrt{3}-1\right)x\right\}\quad;$

$u_2(x) = \sigma_2 \exp\left\{\left(-\sqrt{3}-1\right)x\right\}\quad : (2)$

(3) $\left(D^2+D+\frac{3}{2}\right)u(x) = x\,;\; u_1(x) = \sigma_1 \exp\left(-\frac{x}{2}\right)\sin\left(x\frac{\sqrt{5}}{2}\right)\,;$

$u_2(x) = \sigma_2 \exp\left(-\frac{x}{2}\right)\cos\left(x\frac{\sqrt{5}}{2}\right)\quad : (3)$

(4) $(x^2\,D^2+1)u(x) = \log(x)\,;\; u_1(x) = \sigma_1 \sqrt{x}\,\sin\left[\left(\frac{\sqrt{3}}{2}\right)\log(x)\right]\,;$

$u_2(x) = \sigma_2 \sqrt{x}\,\cos\left[\left(\frac{\sqrt{3}}{2}\right)\log(x)\right]\quad : (4)$

(5) $(x^2\,D^2 - x\,D - 1)u(x) = \frac{1}{x}\,;\; u_1(x) = \sigma_1\, x^{1+\sqrt{2}}\,;\; u_2(x) = \sigma_2 x^{1-\sqrt{2}}\,.$

(6) $(x^2\,D^2 + x\,D + 1)u(x) = x\log(x)\,;$

$u_1(x) = \sigma_1 \sin[\log(x)]\,;\; u_2(x) = \sigma_2 \cos[\log(x)]\quad : (6)$

(7) $(x^2D^2 + x\,D + 3)u(x) = (1/x)\log(x)\,;$

$u_1(x) = \sigma_1 \sin\left[\sqrt{3}\log(x)\right]\,;\; u_2(x) = \sigma_2 \cos\left[\sqrt{3}\log(x)\right]\quad ;\quad (7)$

(8) $[x(x+1)\,D^2 - (x+1)\,D + 1]u(x) = \frac{2}{x}\,;\; u_1(x) = \sigma_1\,(x+1)\,;$

$u_2(x) = \sigma_2[x\log(x+1) + \log(x+1) + 1]\quad : (8)$

(9) $[(x^2+2x)\,D^2 - (x^2+4x+2)\,D + 2(x+1)]u(x) = x+1\,;$

$u_1(x) = \sigma_1\,(x+2)^2\,;\; u_2(x) = \sigma_2 \exp(x)\quad : (9)$

(10) $[(x+1)^2\,D^2 + (x+1)\,D + 1]u(x) = (x+1)\,;$

$u_1(x) = \sigma_1 \sin\left[\log(x+1)\right]\,;\; u_2(x) = \sigma_2 \cos\left[\log(x+1)\right].(10)\quad (7.62)$

7.5 Other Special Situations

Consider the following second-order linear (ODE) with variable coefficients.

$$\left[M_0(x) + M_1(x)\, D + D^2\right] u(x) \;=\; B(x) \quad . \tag{7.63}$$

The complete solution is the sum of complementary solution and the particular integral. The complementary solution of a second-order differential equation with variable coefficients can be very hard to find. Generally, it is constituted of two non-trivial solutions, and assuming neither of these can conveniently be found, one looks for special situations. And there are some such situations, but they require appropriate transformations that use either $M_0(x)$ or $M_1(x)$, functions that occur in (7.63).

7.6 Transformation Using $M_1(x)$ (7.63)

In an attempt to solve (7.63), try a transformation that utilizes $M_1(x)$. Represent the solution of (7.63), namely $u(x)$, as

$$u(x) \;=\; y(x)\, H(x) \quad , \tag{7.64}$$

where

$$H(x) \;=\; \exp\left[\alpha \int M_1(x)\, \mathrm{d}x\right] \quad . \tag{7.65}$$

Inserting (7.64) into (7.63) and doing a little bit of algebra, one can rewrite the latter as

$$\left[\left\{\alpha\,(DM_1) + (M_1)^2(\alpha^2+\alpha) + M_0\right\} + M_1\,(2\alpha+1)\, D + D^2\right] y(x) \;=\; H^{-1}(x)\, B(x) \; . \tag{7.66}$$

If one sets $\alpha = -\frac{1}{2}$, that is, if one uses the relationship

$$(2\alpha+1) \;=\; 0 \quad , \tag{7.67}$$

the term that multiplies D in (7.66) vanishes. And as a result, except for D^2, the only remaining term on the left-hand side of (7.66) is

$$\left\{\alpha\,(DM_1) + (M_1)^2(\alpha^2+\alpha) + M_0\right\} = \left\{-\frac{1}{2}\, DM_1 - \frac{1}{4}\, M_1^2 + M_0\right\} \;\equiv\; G \quad . \tag{7.68}$$

And a happy circumstance is that when G is either equal to a constant, or is proportional to x^{-2}. That is, when

$$\begin{array}{rlll} Either & G = C & , \quad : & (\alpha) \\ Or & G = C\, x^{-2} & , \quad : & (\beta) \end{array} \tag{7.69}$$

where C is a constant.

Examine precisely what all this implies. Recall also that (7.64) specifies $y(x) = H^{-1}\, u(x)$ and with the use of (7.67), (7.65) gives $H^{-1} = \exp\left[\frac{1}{2}\int M_1(x)\,\mathrm{d}x\right]$. Therefore, if in addition to (7.67) one uses also (7.69)-(α), the differential equation for $y(x)$, namely (7.66), becomes

$$\left[C + D^2\right] y(x) = H^{-1}(x)\, B(x)\,, \tag{7.70}$$

On the other hand, when one uses (7.69)-(β) the differential equation (7.66) leads to

$$\left[C\, x^{-2} + D^2\right] y(x) = H^{-1}(x)\, B(x)\,. \tag{7.71}$$

Note, both these equations for $y(x)$—namely (7.70) and (7.71)—would be much easier to solve than the original differential equation (7.63).

7.6.1 Examples Group VIII

A set of second-order differential equations are worked out below.

Reminder: These equations are written in the form of (7.63). So their solution is represented as $u(x)$. By using, (7.63)–(7.71) one calculates first the function $y(x)$. And because $u(x) = y(x)\, H(x)$, after this calculation one multiplies the result by $H(x) = \exp\left[\alpha \int M_1(x)\,\mathrm{d}x\right] = \exp\left[-\frac{1}{2}\int M_1(x)\,\mathrm{d}x\right]$ where $\alpha = -\frac{1}{2}$.

(1) $\left[(4x^2 - 2) - 4x\, D + D^2\right] u \;=\; \exp(x^2)\, x \quad .$

(2) $\left[\left(x^2 - 1\right) - (2x)\, D + D^2\right] u \;=\; x\, \exp\left(\dfrac{x^2 + 2x}{2}\right) \quad .$

(3) $\left[\left(\dfrac{x^2}{4}\right) - x D + D^2\right] u \;=\; \left(\dfrac{x}{8}\right) \exp\left(\dfrac{x^2}{4}\right) \quad .$

(4) $\left[\left(\dfrac{2}{x^2}\right) - \left(\dfrac{2}{x}\right) D + D^2\right] u \;=\; x\, \exp(x) \quad .$

(5) $\left[4(x-1)^2 - 4(x-1) D + D^2\right] u \;=\; \exp(x^2) \quad .$

(6) $\left[\dfrac{2}{(x+2)^2} - \left(\dfrac{2}{x+2}\right) D + D^2\right] u \;=\; x \quad .$

(7) $\left[-\dfrac{1}{4\,x^2} + \left(\dfrac{1}{x}\right) D + D^2\right] u \;=\; x^{-2} \quad .$

(8) $\left[4\, \exp(2\,x) - \{1 + 4\exp(x)\}\, D + D^2\right] u \;=\; \exp(2\,x) \quad .$

$$(9) \quad \left[-\left(\frac{1}{x^2}\right)+\left(\frac{2}{x}\right)D+D^2\right]u \;=\; x^{-1} \quad .$$

$$(10) \quad \left[\left(\frac{4}{x^2}\right)-\left(\frac{4}{x}\right)D+D^2\right]u \;=\; x^3 \quad . \tag{7.72}$$

7.6.2 Solution

(1) $M_0 = \left(4x^2-2\right)$; $M_1 = -4\,x$; $G = C = 0$; $H = \exp(x^2)$;
$B \;=\; \exp(x^2)\,x$.

Therefore,

$$\begin{aligned}\left[C+D^2\right]y \;&=\; \left[0+D^2\right]y = H^{-1}\,B \\ &=\; D^2\,y = x \quad .\end{aligned} \tag{7.73}$$

This is a simple differential equation that can directly be integrated.

$$y(x) = \sigma_1 x + \sigma_2 + \frac{x^3}{6} \quad . \tag{7.74}$$

Therefore, the solution of the differential equation (7.72)-(1) is

$$\boxed{(1)\; u(x) = y\,H = y\,\exp(x^2) \;=\; \left[\sigma_1 x + \sigma_2 + \frac{x^3}{6}\right]\exp(x^2)\,.} \tag{7.75}$$

$$(2) \quad M_0 = \left(x^2-1\right) \;;\; M_1 = -2\,x \;;\; G = C = 0 \;;\; H = \exp\left(\frac{x^2}{2}\right) \;;$$
$$B \;=\; x\,\exp\left(\frac{x^2+2x}{2}\right) \;;\; D^2 y \;=\; H^{-1}\,B \;=\; x\,\exp(x)\,. \tag{7.76}$$

Again, one can proceed by direct integration.

$$y = \sigma_1 x + \sigma_2 + (x-2)\exp(x)\,.$$

And the solution of (7.72)-(2) is:

$$(2)\ u = y\,H = y\,\exp\left(\frac{x^2}{2}\right) \;=\; \{\sigma_1 x + \sigma_2 + (x-2)\exp(x)\}\,\exp\left(\frac{x^2}{2}\right). \tag{7.77}$$

$$(3)\quad M_0 = \frac{x^2}{4}\;;\; M_1 = -x\;;\; G = C = \frac{1}{2}\;;\; H = \exp\left(\frac{x^2}{4}\right)\;;$$
$$B = \exp\left(\frac{x^2}{4}\right)\frac{x}{8}\;;\; \left[\frac{1}{2} + D^2\right] y \;=\; H^{-1}\,G \;=\; \frac{x}{8}\,. \tag{7.78}$$

Differential equation (7.78) is both simple, has constant coefficients, and can readily be solved by using the technique described earlier in (3.5)–(3.73), etc.

$$y = \sigma_1 \sin\left(\frac{x}{\sqrt{2}}\right) + \sigma_2 \cos\left(\frac{x}{\sqrt{2}}\right) + \frac{x}{4}\,. \tag{7.79}$$

The solution is:

$$(3)\ u = y\,H = \left\{\sigma_1 \sin\left(\frac{x}{\sqrt{2}}\right) + \sigma_2 \cos\left(\frac{x}{\sqrt{2}}\right) + \frac{x}{4}\right\}\exp\left(\frac{x^2}{4}\right). \tag{7.80}$$

$$(4)\quad M_0 = \frac{2}{x^2}\;;\; M_1 = -\frac{2}{x}\;;\; G = C = 0\;;\; H = x\;;$$
$$B \;=\; \exp(x)\;x\;;\; D^2 y \;=\; H^{-1}\,B \;=\; \exp(x)\;. \tag{7.81}$$

Again, one can directly integrate this equation.

$$y = \sigma_1\,x + \sigma_2 + \exp(x). \tag{7.82}$$

And the solution is:

$$(4)\ u = y\,H = \{\sigma_1\,x + \sigma_2 + \exp(x)\}\;x\;\;. \tag{7.83}$$

$$(5)\quad M_0 = 4(x-1)^2\;;\; M_1 = -4(x-1)\;;\; G = C = 2\;;\; H = \exp(x^2 - 2x)\;;$$
$$B \;=\; \exp\left(x^2\right)\;;\; [2 + D^2]y \;=\; H^{-1}\,B \;=\; \exp(2x)\;.$$

As before, using the procedure described in (3.5)–(3.73) etc., one gets

$$y = \left\{\sigma_1 \sin\left(x\sqrt{2}\right) + \sigma_2 \cos\left(x\sqrt{2}\right)\right\} + \frac{1}{6}\exp(2x). \tag{7.84}$$

The solution is

$$\begin{aligned}(5)\ u &= y\,H \\ &= \left[\left\{\sigma_1 \sin\left(x\sqrt{2}\right) + \sigma_2 \cos\left(x\sqrt{2}\right)\right\} + \frac{1}{6}\exp(2x)\right]\exp(x^2 - 2x) \\ &= \left\{\sigma_1 \sin\left(x\sqrt{2}\right) + \sigma_2 \cos\left(x\sqrt{2}\right)\right\}\exp(x^2 - 2x) + \frac{1}{6}\exp(x^2)\,.\end{aligned} \tag{7.85}$$

$$\begin{aligned}(6)\quad M_0 = \frac{2}{(x+2)^2}\ ;\ M_1 = -\frac{2}{(x+2)}\ ;\ G = C = 0\ ;\ H = (x+2)\ ; \\ B\ =\ x\ ;\ D^2 y\ =\ H^{-1}\,B\ =\ \frac{x}{x+2}\ .\end{aligned} \tag{7.86}$$

Direct integration yields

$$y = \sigma_1\, x + \sigma_2 + \frac{x^2}{2} - 2(x+2)\log(x+2). \tag{7.87}$$

Therefore,

$$(6)\ u = y\,H = \sigma_3\, x + \sigma_4\, x^2 + \frac{x^3}{2} - 2(x+2)^2 \log(x+2)\,. \tag{7.88}$$

$$\begin{aligned}(7)\quad M_0 = -\left(\frac{1}{4\,x^2}\right)\ ;\ M_1 = \frac{1}{x}\ ;\ G = C = 0\ ;\ H = (x)^{-\frac{1}{2}}\ ; \\ B\ =\ x^{-2}\ ;\ D^2 y\ =\ H^{-1}\,B\ =\ x^{-\frac{3}{2}}\ .\end{aligned} \tag{7.89}$$

Direct integration leads to

$$y = \sigma_1\, x + \sigma_2 - 4\sqrt{x}\,. \tag{7.90}$$

Hence, the solution:

$$(7)\ u = y\,H = [\sigma_1\, x + \sigma_2 - 4\sqrt{x}](x)^{-\frac{1}{2}} = \sigma_1\,(x)^{\frac{1}{2}} + \sigma_2 (x)^{-\frac{1}{2}} - 4\,. \tag{7.91}$$

$$\begin{aligned}(8)\quad M_0\ &=\ 4\,\exp(2\,x)\ ;\ M_1\ =\ \{1 + 4\exp(x)\}\ ;\ G = C\ =\ \frac{-1}{4}\ ; \\ H\ &=\ \exp\left[\frac{x}{2} + 2\exp(x)\right]\ ;\ B\ =\ \exp(2x)\ ; \\ &\left[\frac{-1}{4} + D^2\right] y\ =\ H^{-1}\,B\ =\ \exp\left\{3\left(\frac{x}{2}\right) - 2\exp(x)\right\}\,.\end{aligned} \tag{7.92}$$

Direct integration gives

$$y = \sigma_1 \exp(x/2) + \sigma_2 \exp(-x/2) + \frac{1}{4}\exp\left[-\frac{x}{2} - 2\exp(x)\right]. \tag{7.93}$$

And the solution is

$$\boxed{(8)\ u = yH = \sigma_1 \exp\left[x + 2\exp(x)\right] + \sigma_2 \exp\left[2\exp(x)\right] + \frac{1}{4}.} \tag{7.94}$$

$$\begin{aligned} (9)\quad M_0 &= -\left(\frac{1}{x^2}\right);\ M_1 = \left(\frac{2}{x}\right);\ G = C = \left(\frac{-1}{x^2}\right);\\ H &= \left(\frac{1}{x}\right);\ B = \left(\frac{1}{x}\right);\ \left[\frac{-1}{x^2} + D^2\right] y = H^{-1} B = 1. \end{aligned} \tag{7.95}$$

As before, using the procedure described in (3.5)–(3.73) etc., one gets

$$y = \sigma_1 x^{\frac{\sqrt{5}+1}{2}} + \sigma_2 x^{\frac{-\sqrt{5}+1}{2}} + x^2. \tag{7.96}$$

Therefore, solution to the second-order differential equation (7.72)-(9) with variable coefficients is the following.

$$\boxed{(9)\ u = yH = \sigma_1 x^{\frac{\sqrt{5}-1}{2}} + \sigma_2 x^{\frac{-\sqrt{5}-1}{2}} + x\ .} \tag{7.97}$$

$$\begin{aligned} (10)\quad M_0 &= \left(\frac{4}{x^2}\right);\ M_1 = -\left(\frac{4}{x}\right);\ G = C = \left(\frac{-2}{x^2}\right);\\ H &= x^2;\ B = x^3;\ \left[\frac{-2}{x^2} + D^2\right] y = H^{-1} B = x. \end{aligned} \tag{7.98}$$

Finally, one arrives at

$$y = \sigma_1 x^2 + \frac{\sigma_2}{x} + \frac{x^3}{4}. \tag{7.99}$$

Accordingly, the solution of differential equation (7.72)-(10) is

$$\boxed{(10)\ u = yH = \left[\sigma_1 x^2 + \frac{\sigma_2}{x} + \frac{x^3}{4}\right] x^2.} \tag{7.100}$$

7.6.3 Problems Group V

Use the procedure outlined in (7.64)–(7.71) and determine the differential equation obeyed by $y(x)$. Solve it and calculate $u(x)$.

$$(1) \quad \left[(4x^2+3) - 4x\,D + D^2\right] u \;=\; \exp(x^2) \;.$$

$$(2) \quad \left[x^2 - 2x\,D + D^2\right] u \;=\; \exp\left(\frac{x^2}{2}\right) \;.$$

$$(3) \quad \left[\frac{9}{4}x^2 - 3xD + D^2\right] u \;=\; \exp\left(\frac{3}{4}x^2\right) \;.$$

$$(4) \quad \left[\left(\frac{1}{x^2}\right) - \left(\frac{2}{x}\right) D + D^2\right] u \;=\; x^{-2} \;.$$

7.7 Transformation Using $M_0(x)$

Recall (7.63) and (7.64). They are

$$\begin{aligned} & \left[M_0(x) + M_1(x)\,D + D^2\right] u(x) \;=\; B(x) \;, \\ \textit{and}\;, \qquad & u(x) \;=\; y(x)\,H(x) \;. \end{aligned} \tag{7.101}$$

Unlike (7.65), here $H(x)$ has been set equal to 1 so that

$$\begin{aligned} & u(x) \;\equiv\; y(x) \;, \\ \textit{and}\;, \qquad & \left[M_0(x) + M_1(x)\,D + D^2\right] y(x) \;=\; B(x) \;. \end{aligned} \tag{7.102}$$

Let us introduce a new independent variable, z, defined by the relationship

$$\frac{\mathrm{d}z}{\mathrm{d}x} \;=\; \sqrt{(\pm)M_0} \;. \tag{7.103}$$

In order that $\sqrt{(\pm)M_0}$ is real, the sign $(\pm)$ is so chosen that $(\pm)M_0$ is positive. Consider the following relationships.

$$Dy \;=\; \frac{\mathrm{d}y}{\mathrm{d}x} \;=\; \left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)\left[\frac{\mathrm{d}z}{\mathrm{d}x}\right] \;; \tag{7.104}$$

$$
\begin{aligned}
D^2y = \frac{d}{\mathrm{d}x}[Dy] &= \frac{d}{\mathrm{d}x}\left\{\left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)\left[\frac{\mathrm{d}z}{\mathrm{d}x}\right]\right\} = \frac{d}{\mathrm{d}z}\left\{\left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)\left[\frac{\mathrm{d}z}{\mathrm{d}x}\right]\right\}\frac{\mathrm{d}z}{\mathrm{d}x} \\
&= \left\{\left(\frac{d^2y}{\mathrm{d}z^2}\right)\left[\frac{\mathrm{d}z}{\mathrm{d}x}\right] + \left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)\cdot\frac{d}{\mathrm{d}z}\left[\frac{\mathrm{d}z}{\mathrm{d}x}\right]\right\}\frac{\mathrm{d}z}{\mathrm{d}x} \\
&= \left(\frac{d^2y}{\mathrm{d}z^2}\right)\left[\frac{\mathrm{d}z}{\mathrm{d}x}\right]^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)\cdot\frac{d}{\mathrm{d}z}\left[\frac{\mathrm{d}z}{\mathrm{d}x}\right]\frac{\mathrm{d}z}{\mathrm{d}x} \\
&= \left(\frac{d^2y}{\mathrm{d}z^2}\right)\left[\frac{\mathrm{d}z}{\mathrm{d}x}\right]^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)\cdot\frac{d}{\mathrm{d}x}\left[\frac{dz}{\mathrm{d}x}\right] \\
&= \left(\frac{d^2y}{\mathrm{d}z^2}\right)\left[\frac{\mathrm{d}z}{\mathrm{d}x}\right]^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)\cdot\left[\frac{d^2z}{\mathrm{d}x^2}\right] \quad . \qquad (7.105)
\end{aligned}
$$

Inserting this information into (7.102) leads to

$$
(\pm)y + K(\pm)\frac{\mathrm{d}y}{\mathrm{d}z} + \frac{d^2y}{\mathrm{d}z^2} = \frac{B(x)}{(\pm)M_0} \quad , \qquad (7.106)
$$

where

$$
K(\pm) = \left\{\frac{\frac{d\sqrt{(\pm)M_0}}{\mathrm{d}x} + M_1\sqrt{(\pm)M_0}}{(\pm)M_0}\right\} \quad . \qquad (7.107)
$$

Should $K(\pm)$ turn out to be equal to a constant then (7.106) would be a second-order linear ordinary differential equation with constant coefficients and as such would be easy to solve. And, indeed, if this constant should happen to be zero, the solution would be even easier to obtain.

7.8 Examples Group IX

Work out the following second-order differential equations.

$$
\begin{aligned}
&(1) \quad \left[-\cos^2(x) + \tan(x)\,D + D^2\right]u = -\sin(x)\cos^2(x) \quad . \\
&(2) \quad \left[12\exp(4x) - 2\,(1 + 4\exp(2x))\,D + D^2\right]u = 4\,\exp\left(2\exp(2x) + 4x\right) \quad . \\
&(3) \quad \left[\left(\frac{2}{x^4}\right) + \left(\frac{2}{x}\right)D + D^2\right]u = 2\left(x^{-4} + x^{-6}\right) \quad . \\
&(4) \quad \left[-\left(\frac{1}{x^2}\right) + \left(\frac{1}{x}\right)D + D^2\right]u = x \quad . \qquad (7.108)
\end{aligned}
$$

7.8.1 Solution

Equation (7.108)-(1)

In (7.108)-(1), currently, we have $M_0 = -\cos^2(x)$, but instead of M_0 we choose $(\pm)M_0$ which is required to be $+\cos^2(x)$. Therefore, in (1) one must set $(\pm) = -1$. One follows the same procedure in equations (2) $\rightarrow$ (4).

$$\begin{aligned} &Here\ ,\quad (\pm) \equiv (-)\ . \\ &Also\ :\ \frac{\mathrm{d}z}{\mathrm{d}x} = \sqrt{(\pm)M_0} = \cos(x)\ ;\ z = \sin(x)\ ;\ M_1 = \tan(x)\ . \\ &\frac{d\sqrt{(\pm)M_0}}{\mathrm{d}x} = -\sin(x)\ ;\ M_1\sqrt{(\pm)M_0} = \sin(x)\ ;\ K(\pm) = 0\ ; \\ &B(x) = -\sin(x)\,\cos^2(x)\ . \end{aligned} \tag{7.109}$$

Accordingly, via the use of (7.103)–(7.107), (7.108)-(1) is reduced to $(\pm)y + \frac{d^2y}{\mathrm{d}z^2} = \frac{B(x)}{(\pm)M_0}$. Or equivalently

$$\begin{aligned} -y + \frac{d^2y}{\mathrm{d}z^2} &= \frac{-\sin(x)\,\cos^2(x)}{\cos^2(x)} = -\sin(x) \\ &= -z\ . \end{aligned} \tag{7.110}$$

This equation has constant coefficients and is readily solved by using the techniques described in a previous chapter. Its solution is:

$$(1):\quad y = \sigma_1\,\exp(z) + \sigma_2\,\exp(-z) + z\ . \tag{7.111}$$

One can change back to the original variable x by substituting $\sin(x)$ for z. Therefore, the solution of (7.108)-(1) is:

$$\boxed{(1):\quad y = \sigma_1\,\exp[\sin(x)] + \sigma_2\,\exp[-\sin(x)] + \sin(x)\ .} \tag{7.112}$$

Equation (7.108)-(2)

In (7.108)-(2), M_0 is positive. So one can safely ignore the use of the symbol $(\pm)$.

$$\begin{aligned} &(2)\quad M_0 = 12\exp(4x)\ ;\ \frac{\mathrm{d}z}{\mathrm{d}x} = \sqrt{M_0} = \sqrt{12}\,\exp(2x)\ ;\ z = \sqrt{3}\,\exp(2x)\ ; \\ &M_1 = -2[1 + 4\exp(2x)]\ ;\ \frac{d\sqrt{M_0}}{\mathrm{d}x} = \sqrt{48}\,\exp(2x)\ ; \\ &M_1\sqrt{M_0} = -\sqrt{48}\,[\exp(2x) + 4\exp(4x)]\ ;\ K = -\left(\frac{4}{\sqrt{3}}\right)\ ; \\ &\frac{B(x)}{M_0} = \left(\frac{1}{3}\right)\exp\left[2\exp(2x)\right] = \left(\frac{1}{3}\right)\exp\left[\frac{2z}{\sqrt{3}}\right]\ . \end{aligned} \tag{7.113}$$

Accordingly, via the use of (7.103)–(7.106), (7.108)-(2) is reduced to

$$y - \left(\frac{4}{\sqrt{3}}\right)\frac{\mathrm{d}y}{\mathrm{d}z} + \frac{d^2y}{\mathrm{d}z^2} \;=\; \frac{B(x)}{M_0} \;=\; \left(\frac{1}{3}\right)\exp\left[\frac{2z}{\sqrt{3}}\right] . \tag{7.114}$$

This equation has constant coefficients and is readily solved by using the techniques described in a previous chapter. Its solution is:

$$(2): \quad y \;= \sigma_1\, \exp\left(\frac{z}{\sqrt{3}}\right) + \sigma_2\, \exp\left(z\sqrt{3}\right) - \left(\frac{2z}{\sqrt{3}}\right) . \tag{7.115}$$

One can change back to the original variable x by substituting $\sqrt{3}\exp(2x)$ for z. Therefore, the solution of (7.108)-(2) is

$$(2): \quad y \;= \sigma_1\, \exp[\exp(2x)] + \sigma_2\, \exp[3\exp(2x)] - \exp[2\exp(2x)] \;. \tag{7.116}$$

Equation (7.108)-(3)

(3): Next consider (7.108)-(3). Here too, M_0 is positive. So the symbol $(\pm)$ is not needed.

$$\begin{aligned}
(3): \quad M_0 \;&= \frac{2}{x^4} \;;\; \frac{\mathrm{d}z}{\mathrm{d}x} \;= \sqrt{M_0} \;= \frac{\sqrt{2}}{x^2} \;;\; z \;= \; -\frac{\sqrt{2}}{x} \;; \\
M_1 \;&= \frac{2}{x} \;;\; \frac{d\sqrt{M_0}}{\mathrm{d}x} \;= \; -\frac{\sqrt{8}}{x^3} \;; \\
M_1\sqrt{M_0} \;&= \frac{\sqrt{8}}{x^3} \;;\; K \;= 0 \;;\; x \;= \; -\frac{\sqrt{2}}{z} \;. \\
\frac{B}{M_0} \;&= \frac{2\,(x^{-4} + x^{-6})}{2(x^{-4})} \;= 1 + x^{-2} \;= \; 1 + \frac{z^2}{2} \;.
\end{aligned} \tag{7.117}$$

Accordingly, via the use of (7.103)–(7.106), (7.108)-3 is reduced to

$$y + \frac{d^2y}{\mathrm{d}z^2} \;=\; \frac{B}{M_0} \;= 1 + \frac{z^2}{2} \quad . \tag{7.118}$$

This equation is readily solved.

$$(3): \quad y \;= \; -\sigma_1\, \sin(z) + \sigma_2\, \cos(z) + \frac{z^2}{2} \;. \tag{7.119}$$

Again, one can change back to the original variable by substituting $\left(-\frac{\sqrt{2}}{z}\right)$ for z. And the solution of (7.108)-(3) is

$$(3): \quad y(x) = \sigma_1 \sin\left(\frac{\sqrt{2}}{x}\right) + \sigma_2 \cos(\left(\frac{\sqrt{2}}{x}\right) + \frac{1}{x^2} \ . \tag{7.120}$$

Equation (7.108)-(4)

(4): In (7.108)-(4), M_0 is equal to $\frac{-1}{x^2}$. Therefore, as suggested earlier, we choose $(\pm) \equiv -$.

$$\begin{aligned} (4): \quad & (\pm) M_0 = \frac{1}{x^2} \ ; \ \frac{\mathrm{d}z}{\mathrm{d}x} = \sqrt{(\pm) M_0} = \frac{1}{x} \ ; \ Therefore, \ z = \log x \ ; \\ Also, \quad & M_1 = \frac{1}{x} \ ; \ \frac{d\sqrt{(\pm) M_0}}{\mathrm{d}x} = -\frac{1}{x^2} \ ; \\ M_1 \sqrt{\pm M_0} \quad & = \frac{1}{x^2} \ ; \ K(\pm) = 0 \ ; \ B = x \ . \end{aligned} \tag{7.121}$$

Accordingly, via the use of (7.103)–(7.107), (7.108)-(4) is reduced to

$$\begin{aligned} (\pm) y + \frac{d^2 y}{\mathrm{d}z^2} &= \frac{B}{(\pm) M_0} \ , \\ -y + \frac{d^2 y}{\mathrm{d}z^2} &= \frac{x}{\left(\frac{1}{x^2}\right)} = x^3 = \exp(3\,z) \ . \end{aligned} \tag{7.122}$$

This equation has constant coefficient and is solved by previously described techniques. Its solution is

$$y = \sigma_1 \exp(z) + \sigma_2 \exp(-z) + \frac{\exp(3\,z)}{8} \ . \tag{7.123}$$

Now change back to the original variable x by substituting $\log(x)$ for z in (7.123). The result is the desired solution to (7.108)-(4).

$$(4): y(x) = \sigma_1 x + \frac{\sigma_2}{x} + \exp(-z) + \frac{x^3}{8}. \tag{7.124}$$

Chapter 8
Oscillatory Motion



Oscillatory motion is central to the description of acoustics and the effects of inter-particle interaction in many physical systems. In its most accessible form, oscillatory motion is simple harmonic. Such motion—which in this chapter is described first—has a long and distinguished history of use in modeling physical phenomena .

Described next is anharmonic motion which somewhat more realistically represents the observed behavior of oscillatory physical systems. To this end, a detailed analysis of transient state motion is presented for a point mass for two different oscillatory systems. These are as follows:

(1) The point mass, m, is tied to the right end of a long, massless coil spring placed horizontally in the x-direction on top of a long, level table. The left end of the coil is fixed to the left end of the long table. The motion of the mass is slowed by frictional force, that is, proportional to its momentum $m\,v(t)$. In its completely relaxed state, the spring is in equilibrium and the mass is in its equilibrium position (EP).

(2) Because the differential equations needed for analyzing damped oscillating pendula are prototypical of those used in the studies of electromagnetism, acoustics, mechanics, chemical and biological sciences, and engineering, we analyze next a pendulum consisting of a (point-sized) bob of mass m, that is, tied to the end of a massless stiff rod of length l. The rod hangs down, in the negative z-direction, from a hook that has been nailed to the ceiling. The pendulum is set to oscillate in two-dimensional motion in the $x - z$ plane. Air resistance is approximated as a frictional force proportional to the speed with which the bob is moving. The ensuing friction slows the oscillatory motion.

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8.1 Periodic Functions

A function $F(x)$ is periodic if for all values of x there exists a positive constant, CONST, such that

$$F(x + \text{CONST}) = F(x) \ . \tag{8.1}$$

The period of the function is the smallest positive value of CONST, say, equal to $2P$ so that

$$F(x + 2P) = F(x) \ . \tag{8.2}$$

Beauty and simplicity of oscillatory periodic motion has long attracted scientists interested in modeling observed physical phenomena.

The simplest possible periodic motion is called simple harmonic motion(s.h). (s.h) is force-free, undamped periodic motion with constant period of oscillation and unchanging size of the maximum and the minimum displacements.

Oscillatory Motion of Mass Tied to Spring on a Table

Consider, for instance, a point mass, m, tied to the right end of a long massless coil spring placed horizontally in the x-direction on top of a long, level, friction-free smooth table. The left end of the coil is fixed to the left end of the long table. In its completely relaxed state, the spring is in equilibrium and the mass, tied to the right end of the coil, is in its equilibrium position. The equilibrium position is henceforth to be referred to as the (EP).

Pull the mass away from the (EP) by distance $+x$. The extension of the spring beyond the (EP) provides a restoring force. In general, the strength of the restoring force, $F(x)$, is a complicated function of the extension x. If $F(x)$ is analytic within $-X_0 \leq x \leq X_0$, the $Maclaurin - Taylor$[27.–28.] series expansion obtains.

$$F(x) = F(0) + \sum_{i=1}^{\infty} \left(\frac{\mathrm{d}^i F}{\mathrm{d}x^i} \right) x^i \ . \tag{8.3}$$

Because there is no restoring force at zero extension, $F(0)$ must be vanishing. Also, for small extension $|x| < |X_0|$— where $|X_0|$ is the maximum extension where the Hooke's law accurately holds—the Hooke' law[29.] applies. That is

$$F(x) \approx -(m\,K)\,x \tag{8.4}$$

where $(m\,K) = -\left(\frac{\mathrm{d}F}{\mathrm{d}x}\right)$ is the so-called Hooke's constant. If the spring is extended toward the positive direction—meaning, if x is positive—the restoring force must be toward the negative direction. Thus the Hooke's constant $(m\,K)$ is positive.

In practice, the restoring force may not be linear. As such, there would be anharmonicity related to nonlinear contribution to $F(x)$. Also, many oscillatory processes encounter damping due to a variety of frictional effects. Yet, the restoring force $F(x)$, for moderate to small displacement, is often well approximated by Hooke's law.

8.1.1 Harmonic Oscillation

The Newtonian equation of motion of the mass m that itself is constant—meaning, $\frac{\mathrm{d}m}{\mathrm{d}t} = 0$—and experiences a force $F(x)$ is

$$F(x) \;=\; m\,\frac{\mathrm{d}^2x}{\mathrm{d}t^2} \;=\; m\,D^2x \;\;. \tag{8.5}$$

Using (8.4) and (8.5), the resultant second-order differential equation (relevant only for small $|x|$) is

$$m\,[D^2 + {\Omega_0}^2]x = 0 \;\;, \tag{8.6}$$

where

$$D \;=\; \frac{\mathrm{d}}{\mathrm{d}x} \;\;;\;\; D^n \;=\; \frac{\mathrm{d}^n}{\mathrm{d}x^n} \;\;;\;\; {\Omega_0}^2 = K \;\;. \tag{8.7}$$

(Note: Ω_0 is real because K is positive.)

The second-order differential equation (8.6) has constant coefficients. Therefore, it is readily solved by using the standard rules described in (3.5) to (3.42). The solution is sinusoidal

$$x(t) \;=\; \sigma_0\,\cos\,(\Omega_0 t + \phi_0) \tag{8.8}$$

and has three parameters: σ_0, Ω_0, and ϕ_0. To determine them, proceed as follows.

The maximum value of cos(...) is = 1 making σ_0 the maximum displacement of the point mass beyond the (EP).

The period of the sinusoidal oscillatory harmonic motion is defined to be the minimum time, τ, and it takes to exactly complete one whole cycle. That is

$$x(t+\tau) \;=\; x(t)$$

so that

$$\cos\,[\Omega_0(t+\tau) + \phi_0] \;=\; \cos\,(\Omega_0 t + \phi_0) \;\;. \tag{8.9}$$

This requires $\Omega_0\,\tau$ to equal 2π.

$$\tau = \frac{2\pi}{\Omega_0} = \frac{2\pi}{\sqrt{K}} \ . \tag{8.10}$$

The frequency, ν, of the sinusoidal oscillation is

$$\nu = \frac{1}{\tau} \ . \tag{8.11}$$

The angular frequency of the oscillations is defined to be $2\pi\nu$.

$$2\pi\nu = \frac{2\,\pi}{\tau} = \Omega_0 = \sqrt{K} \ . \tag{8.12}$$

The one remaining parameter, namely the phase angle ϕ_0, can be determined from one additional piece of information, e.g., either the position or the velocity at some specific time. For instance, assume the mass is at its maximum displacement—that is, σ_0—at time $t = 0$. Then, according to (8.8)

$$\cos\left(\phi_0\right) = 1 \tag{8.13}$$

meaning $\phi_0 = 0$. One could have got the same result also from requiring at maximum displacement, σ_0, the velocity is vanishing. That is, at $t = 0$

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)_{t=0} = -\,\sigma_0\,\omega_0\,\sin\left[\omega_0\,t + \phi_0\right] = -\,\sigma_0\,\omega_0\,\sin\left[\phi_0\right] = 0 \ . \tag{8.14}$$

Thus, ϕ_0 is indeed zero, and the solution of the differential equation (8.6) is

$$x(t) = \sigma_0\,\cos\left(\Omega_0 t\right) \ . \tag{8.15}$$

8.1.2 Energy

The (s.h) oscillatory motion described above is infinitely long-lived. How does the system energy fair? The total energy is the sum of the kinetic energy

$$E_{\text{kinetic}} \equiv \frac{m}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 = \frac{m}{2}\left[-x_0\,\Omega_0\,\sin\left(\Omega_0 t\right)\right]^2 = \frac{1}{2}\,m\,K\,\left(x_0\right)^2\,\left[\sin\left(\Omega_0 t\right)\right]^2 ,$$

and the potential energy, which is equal to the work done in extending the spring a distance x

$$E_{\text{potential}} \equiv -\int_0^x F(x)\,\mathrm{d}x = \frac{1}{2}\,m\,K\,x^2 = \frac{1}{2}\,m\,K\,\left(x_0\right)^2\,\left[\cos\left(\Omega_0 t\right)\right]^2 \ .$$

That is

$$E_{\text{total}} = E_{\text{kinetic}} + E_{\text{potential}} = \frac{1}{2} m K (x_0)^2 \left\{[\sin(\Omega_0 t)]^2 + [\cos(\Omega_0 t)]^2\right\}$$
$$= \frac{1}{2} m K (x_0)^2 . \tag{8.16}$$

Clearly, the total energy is constant, independent of time. Notice how at maximum displacement in either direction when $[\cos(\Omega_0 t)]^2 = 1$ and $[\sin(\Omega_0 t)]^2 = 0$ energy is all potential. This happens when $\Omega_0 t = n\,\pi$, n = 0, 1, 2, ... At the midpoint, that is, at the (EP), when $[\cos(\Omega_0 t)]^2 = 0$ and $[\sin(\Omega_0 t)]^2 = 1$ the energy is all kinetic. This happens whenever $\Omega_0 t = (2\,n + 1)\,(\pi/2)$. In between these two extremes, the energy is partially kinetic and partially potential.

8.1.3 Energy Conservation and Equation of Motion

Conservation of total energy provides a convenient tool for determining the differential equation obeyed by the harmonically oscillating mass. Both the velocity, v, and the position, x, in the expression for total energy are time dependent.

$$E_{\text{total}}(t) = E_{\text{kinetic}}(t) + E_{\text{potential}}(t) = \frac{m}{2} v(t)^2 + \frac{1}{2} m K x(t)^2 . \tag{8.17}$$

Yet, given absence of lossy effects and zero exchange of energy with the outside, the total energy must remain unchanged. That is

$$\frac{\mathrm{d}E_{\text{total}}(t)}{\mathrm{d}t} = 0 . \tag{8.18}$$

Using (8.17) for $E_{\text{total}}(t)$, (8.18) gives

$$m\,v(t)\frac{\mathrm{d}v(t)}{\mathrm{d}t} + m\frac{\mathrm{d}x(t)}{\mathrm{d}t} K x(t) = m\,v(t)\left[\frac{\mathrm{d}v(t)}{\mathrm{d}t} + K x(t)\right] = 0 .$$

The relationship $\frac{\mathrm{d}v(t)}{\mathrm{d}t} = \frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2}$ leads to the relevant differential equation.

$$\left[\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} + K x(t)\right] = 0 . \tag{8.19}$$

8.1.4 Anharmonic Damped Motion

Assume the point mass being described is actually quite rough and its motion on the table is subject to frictional force that is proportional to its momentum $m\,v(t)$. No external force is applied, and no other forces affect the motion of the long massless coil. Then, the Newtonian equation of motion of the mass is

$$\begin{aligned} m\,\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} &= -\,\text{Friction} \;-\; \text{Hooke's Restoring Force} \\ &= -(2\,\alpha)\,m\,v(t) \;-\; (m\,K)\,x(t)\;, \end{aligned} \tag{8.20}$$

where $(2\,\alpha)$ is a constant that describes the strength of friction. Dividing by m and using $v(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t}$ leads to the differential equation of damped harmonic motion that is not subject to any external force.

$$\begin{aligned} &\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} + 2\,\alpha\,\frac{\mathrm{d}x(t)}{\mathrm{d}t} + K\,x(t) \\ &\equiv \left[D^2 + 2\,\alpha\,D + K\right]x(t) \;=\; 0\;. \end{aligned} \tag{8.21}$$

Solution

As is the case for all homogeneous linear (ODE), particular integral, I_{pi}, for (8.21) is vanishing. Given that (8.21) has constant coefficients, its characteristic equation, E_{ch}, can be written simply by substituting k for D.

$$k^2 + 2\,\alpha\,k + K = 0 \quad . \tag{8.22}$$

As expected, for a second-order differential equation the E_{ch} is quadratic in the variable k. Its two roots

$$\begin{aligned} k \;&=\; -\,\alpha + \sqrt{\alpha^2 - K} \;\equiv\; -\,k_{01}\;, \\ k \;&=\; -\,\alpha - \sqrt{\alpha^2 - K} \;\equiv\; -\,k_{02}\;, \end{aligned} \tag{8.23}$$

provide the complementary solution $S_{\text{comp}}(t)$.

$$S_{\text{comp}}(t) \;\rightarrow\; x(t) = \sigma_1 \exp\left(-k_{01}\,t\right) \;+\; \sigma_2 \exp\left(-k_{02}\,t\right)\;. \tag{8.24}$$

Note, $x(t)$ signifies the location of the mass at time t.

Henceforth, for convenience, we shall work only with $t \geq 0$.

8.1.5 Over-Damped Anharmonic Motion

When

$$\alpha^2 > K \tag{8.25}$$

friction outweighs the Hooke's attraction, and motion is over-damped. Because α is proportional to resistance to motion, α must be > 0. And because Hooke's restoring force, and therefore K, is known to be positive, for over-damped motion the inequality $\alpha > \sqrt{\alpha^2 - K}$ holds. Indeed, according to (8.23), for over-damped motion the following relationships are obtained.

$$\begin{aligned} k_{01} > 0 \quad ; \qquad k_{02} > 0 \ ; \\ (k_{02} - k_{01}) = 2\sqrt{\alpha^2 - K} \ > \ 0 \ . \end{aligned} \tag{8.26}$$

Meaning, for over-damped motion both k_{01} and k_{02} are necessarily positive, and $(k_{02} - k_{01})$ is greater than zero. Therefore, irrespective of whether the constants σ_1 and/or σ_2 are positive or negative in (8.24), the position $x(t \to \infty)$ must tend to zero which is the (EP). But on the other hand, at time $t = 0$, we have

$$x(0) = \sigma_1 \ + \ \sigma_2 \ . \tag{8.27}$$

Should it happen that $x(0)$ is a positive distance away from (EP), then there is a possibility that the point mass would start moving leftward, come to a momentary stop, and reverse direction to head back. But whether it oscillates further, or indeed stays put on the original side never crossing over to the other, depends—see below—on the sign and the relative size of the two constants σ_1 and σ_2. As usual, two boundary conditions are needed to fix the two constants σ_1 and σ_2.

8.1.6 Both σ_1 and σ_2 Positive Mass Stays on Original Side

For over-damped motion, according to (8.26), if both σ_1 and σ_2 are positive the mass will stay on the positive side of the (EP) and never cross over to the other side. Also, with the passage of time the mass will continue monotonically approaching the (EP).

What about its velocity v_t?

$$v_t = \frac{\mathrm{d}x}{\mathrm{d}t} = -\sigma_1 \, k_{01} \ \exp\left(-k_{01} \, t\right) \ - \ \sigma_2 \, k_{02} \ \exp\left(-k_{02} \, t\right) \ . \tag{8.28}$$

As shown in Fig. 8.1, for positive values of the constants σ_1 and σ_2, at $t = 0$ the mass is moving leftward toward the (EP). Its velocity approaches zero as $t \to \infty$, and in this process, the velocity undergoes no extrema.

Confirmation of above statement is given below. Rewrite (8.24).

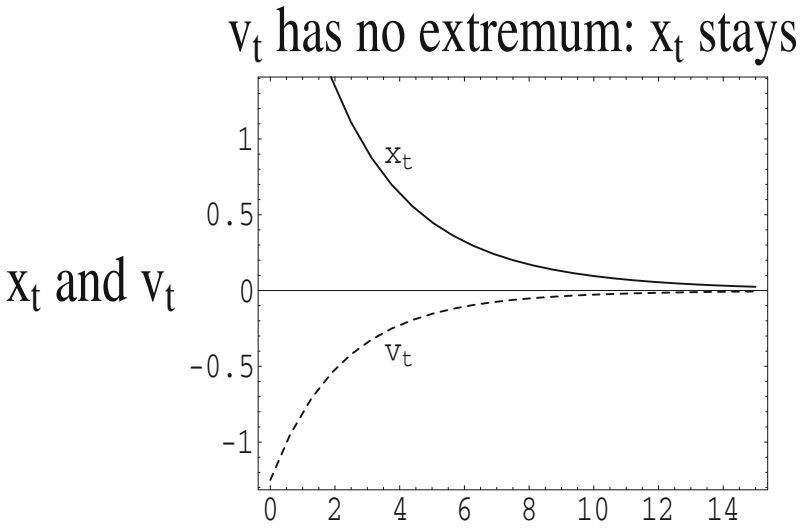


Fig. 8.1 Mass tied to an over-damped spring, with positive values of σ's, stays on the original side

$$x(t) = \sigma_1 \exp(-k_{01}\, t) \left\{ 1 + \left(\frac{\sigma_2}{\sigma_1} \right) \exp[-(k_{02} - k_{01})\, t] \right\} \quad . \tag{8.29}$$

Given σ_1 is a constant, be it positive or negative, the term $\sigma_1 \exp(-k_{01}\, t)$ does not change sign with the passage of time. If there is to be a change in sign of the location $x(t)$, it will have to come from the second term, namely $\left\{ 1 + \left(\frac{\sigma_2}{\sigma_1} \right) \exp[-(k_{02} - k_{01})\, t] \right\}$. At $t = 0$, this term is equal to $1 + \left(\frac{\sigma_2}{\sigma_1} \right)$. And as $t \to \infty$, because $(k_{02} - k_{01}) > 0$, it tends to $+1$. These two numbers have the same sign only if

$$1 + \left(\frac{\sigma_2}{\sigma_1} \right) > 0 \quad . \tag{8.30}$$

As long as

$$\left(\frac{\sigma_2}{\sigma_1} \right) > -1 \tag{8.31}$$

equation (8.30) is satisfied, and the mass stays on the original side of the (EP).

But what about its velocity? And additionally, does the velocity v_t and possibly also the location $x(t)$ undergo any extrema?

Extrema in position $x(t)$—if any—would occur at time $t = t_{\text{x:extreme}}$ such that $\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)_{t=t_{\text{x:extreme}}} = 0$. On the other hand, the velocity extremum—if any—would occur at $t = t_{\text{v:extreme}}$ when $\left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)_{t=t_{\text{v:extreme}}} = \left(\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\right)_{\text{t=v:extreme}} = 0$. In other words, $t_{\text{x:extreme}}$ and $t_{\text{v:extreme}}$ would obey the following relationships:

$$\begin{aligned}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)_{t_{\text{x:extreme}}} &= -\sigma_1\, k_{01}\exp(-k_{01}\, t_{\text{x:extreme}}) - \sigma_2\, k_{02}\exp(-k_{02}\, t_{\text{x:extreme}}) \\ &= -\sigma_1\, k_{01}\exp(-k_{01}\, t_{\text{x:extreme}}) \\ &\left\{1 + \left(\frac{\sigma_2 k_{02}}{\sigma_1 k_{01}}\right)\exp[-(k_{02} - k_{01})\, t_{\text{x:extreme}}]\right\} = 0\,. \end{aligned} \tag{8.32}$$

$$\begin{aligned}\left(\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\right)_{t=t_{\text{v:extreme}}} &= \sigma_1\,(k_{01})^2\exp(-k_{01}\, t_{\text{v:extreme}}) + \sigma_2\,(k_{02})^2\exp(-k_{02}\, t_{\text{v:extreme}}) \\ &= \sigma_1\,(k_{01})^2\exp(-k_{01}\, t_{\text{v:extreme}})) \\ &\left\{1 + \left(\frac{\sigma_2 k_{02}^2}{\sigma_1 k_{01}^2}\right)\exp[-(k_{02} - k_{01})\, t_{\text{v:extreme}}]\right\} = 0\,. \end{aligned} \tag{8.33}$$

Items within {...} in (8.32) and (8.33) add to zero. Therefore

$$\begin{aligned} t_{\text{x:extreme}} &= \left(\frac{1}{k_{02} - k_{01}}\right)\log\left[-\frac{\sigma_2}{\sigma_1}\left(\frac{k_{02}}{k_{01}}\right)\right]\;; \\ t_{\text{v:extreme}} &= \left(\frac{1}{k_{02} - k_{01}}\right)\log\left[-\frac{\sigma_2}{\sigma_1}\left(\frac{k_{02}}{k_{01}}\right)^2\right]\;. \end{aligned} \tag{8.34}$$

Since k_{01} and k_{02} are > 0, if σ_1 and σ_2 have the same sign, $t_{\text{x:extreme}}$ and $t_{\text{v:extreme}}$ will not exist because logarithm of negative quantity is a complex number while the time must necessarily be real. Therefore, under these circumstances $x(t)$ and $v(t)$ will not have extrema.

8.1.7 *Over-Damped Anharmonic Motion $\left(\frac{\sigma_2}{\sigma_1}\right)$ Negative But > -1 Mass Stays on Original Side*

Mass stays on the original side of the origin if (8.30) is satisfied. And this can happen—as demonstrated in Fig. 8.2—even when $\left(\frac{\sigma_2}{\sigma_1}\right)$ is negative as long as it is > -1. Also because the ratio $\left[-\frac{\sigma_2}{\sigma_1}\left(\frac{k_{02}}{k_{01}}\right)\right]$ is then positive, and logarithm of a positive quantity is real, according to (8.34) extrema can occur both for the position x and the velocity v.

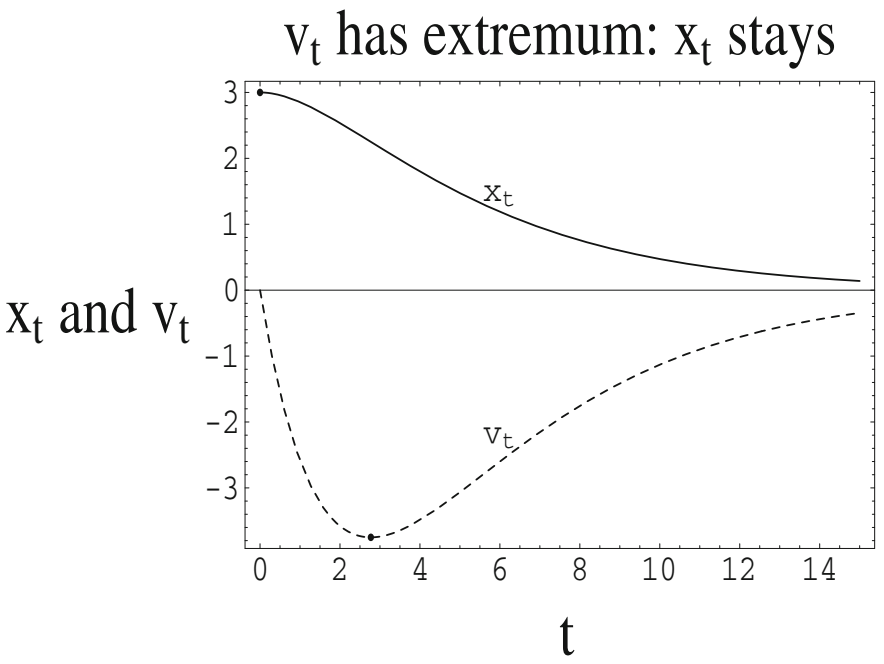


Fig. 8.2 An over-damped spring with positive σ_1 and not too negative σ_2. Mass stays on the original side. At time t, position x_t and (ten times) velocity v_t are displayed as function of t. Dots locate extrema

8.1.8 Over-Damped Anharmonic Motion Equation (8.31) Not-Satisfied Mass Crosses over

Next, consider systems where (8.30) is not satisfied: Meaning the requirement for staying on the original side is violated. One class of such systems are over-damped springs with negative σ_1 and larger positive σ_2. For such systems as t increases from zero the mass moves leftward, stops momentarily, and restarts moving in positive direction to its final position of rest. According to (8.34), extrema in both the position and the velocity are possible—positions indicated by dots. This behavior is portrayed in Fig. 8.3.

8.1.9 Critically Damped Anharmonic Motion

Unlike the case where friction overpowers the Hooke's attraction—for instance, (8.25)—if the force due to Hooke's attraction should happen to be of similar strength to friction, the system would be critically damped. More precisely, this is the case

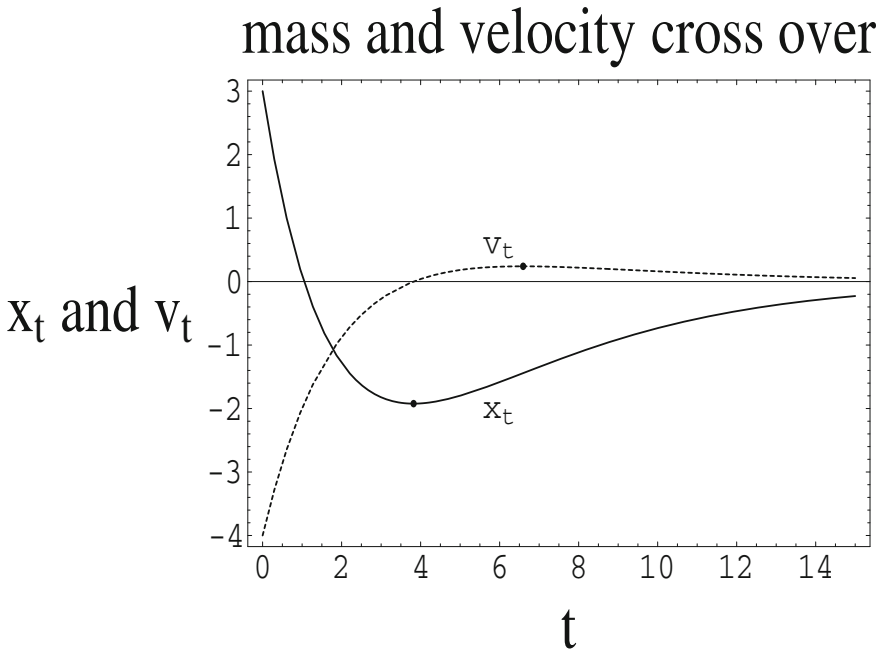


Fig. 8.3 Represented here is the behavior of an over-damped spring, where (8.31) is not satisfied. For $\sigma_1 = -10,\ k_{01} = \frac{1}{4}$, and $\sigma_2 = 13,\ k_{02} = \frac{1}{2}$, $x(t) \equiv x_t$ and $v(t) \equiv v_t$ are displayed as function of t. Dots locate extrema

when $K = \alpha^2$. Then, the characteristic equation, (8.22),

$$k^2 + 2\,\alpha\,k + \alpha^2 \;=\; (k+\alpha)^2 \;=\; 0 \quad ,$$

has two roots,

$$k \;=\; -\,\alpha \quad , \tag{8.35}$$

that are equal. According to well-established procedure, the solution to this differential equation can then be expressed as—[see (3.37)]—

$$x_t \;=\; (\sigma_3 + \sigma_4\,t)\exp(k\,t) \;=\; (\sigma_3 + \sigma_4\,t)\exp(-\alpha\,t) \quad . \tag{8.36}$$

As such

$$v_t \;=\; \frac{\mathrm{d}x_t}{\mathrm{d}t} = (\sigma_4 - \alpha\,\sigma_3 - \alpha\,\sigma_4\,t)\exp\left(-\alpha\,t\right). \tag{8.37}$$

Note a dot indicates the location of the extremum (Fig. 8.4).

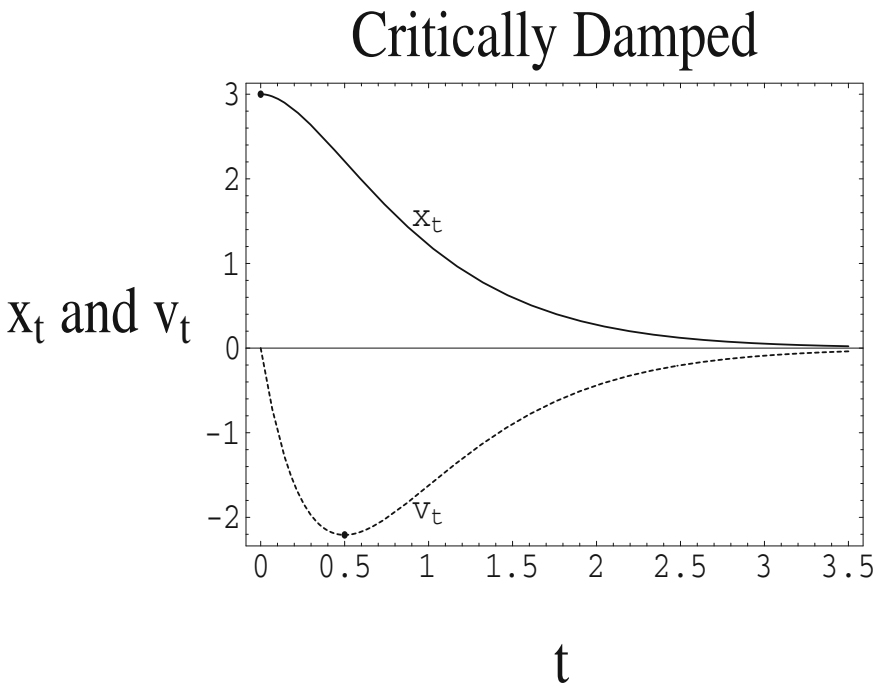


Fig. 8.4 x_t and v_t of a mass being pulled by critically damped spring with $\sigma_1 = 3$ and $\alpha = 2$, are displayed as function of t. Because $\alpha > 0$, x_t remains positive, mass stays on the original positive side. Dots indicate extrema

8.1.10 An Exercise

Assume, much like Fig. 8.2 at time $t = 0$, the spring is extended and the mass is at position x_0 and has just begun moving leftward: meaning $v_0 = -0$. According to (8.36) and (8.37), one has

$$\begin{aligned} v_0 \;=\; -0 = (\sigma_4 - \alpha\,\sigma_3) \;\;, \\ \sigma_4 = \alpha\,\sigma_3 \;\;, \\ x_t = \sigma_3\,(1+\alpha\,t)\,\exp(-\alpha\,t) \;=\; x_0\,(1+\alpha\,t)\,\exp(-\alpha\,t); \\ v_t = \frac{\mathrm{d}x_t}{\mathrm{d}t} \;=\; -\,x_0\,\alpha^2\,t\,\exp(-\alpha\,t)\;; \\ \frac{\mathrm{d}v_t}{\mathrm{d}t} = \frac{\mathrm{d}^2x_t}{\mathrm{d}t^2} \;=\; -\,x_0\,\alpha^2\,\exp(-\alpha\,t)\,[1-\alpha\,t]\;. \end{aligned} \tag{8.38}$$

According to (8.38), x_t remains positive because $\alpha > 0$. Meaning, mass stays on the original positive side. And as t increases, mass approaches exponentially the (EP). At $t = 0$, the first derivative of x_t is vanishing while its second derivative is negative.

Thus, x_0 is a maximum. The velocity starts at zero, moves leftwards, and as $t \to \infty$ it again $\to 0$. In between, the velocity reaches an extremum $-\frac{x_0\,\alpha}{E}$ at time $t = \frac{1}{\alpha}$.

8.1.11 Under-Damped Spring

Sinusoidal Motion

When Hooke's attractive force is stronger than friction, i.e., $K > \alpha^2$, the spring is under-damped. The two roots of the characteristic equation are complex, namely

$$-k_{01} \;=\; -\alpha - i\sqrt{K-\alpha^2}\,; \;\; -k_{02} \;=\; -\alpha + i\sqrt{K-\alpha^2}\,. \tag{8.39}$$

The complementary solution can readily be found by using a procedure similar to that in (3.21)–(3.24).

$$\begin{aligned} S_{\text{comp}} \to x_t \;&=\; \sigma_1 \exp\left(-k_{01}\,t\right) \;+\; \sigma_2 \exp\left(-k_{02}\,t\right)\;, \\ &=\; \exp\left(-\alpha\,t\right)\left[\sigma_3 \sin\left(t\,\sqrt{K-\alpha^2}\right) + \sigma_4 \cos\left(t\,\sqrt{K-\alpha^2}\right)\right]\;, \\ &=\; \sigma_5\, \exp\left(-\alpha\,t\right) \cos\left(t\sqrt{K-\alpha^2} - \phi\,\right)\;, \end{aligned} \tag{8.40}$$

where $\sigma_3 = -i(\sigma_1 - \sigma_2)$, $\sigma_4 = (\sigma_1 + \sigma_2)$, and $\sigma_5 = \left(\sqrt{\sigma_3^2 + \sigma_4^2}\right)$. Also, $\sin\phi = \left(\frac{\sigma_3}{\sigma_5}\right)$ and $\cos\phi \;=\; \left(\frac{\sigma_4}{\sigma_5}\right)$.

The sinusoidal time dependence causes the mass to oscillate back and forth across the origin. And it does so, in principle, an infinite number of times. However, because the size of the oscillations decreases exponentially, the oscillations become all but invisible—depending on how large α is—long before their theoretical end.

8.2 Oscillating Pendulum

Oscillatory motion is central to understanding inter-particle interaction in many physical systems. Furthermore, the differential equations needed for analyzing damped externally driven oscillating pendula are prototypical of those used in studies of electromagnetism, acoustics, mechanics, chemical and biological sciences, and engineering.

We have already treated briefly the oscillatory motion of a spring on a table. More informative is the motion of a pendulum—see Fig. 8.5—consisting of a (point-sized) bob of mass m, that is, tied to the end of a massless stiff rod of length l. The rod hangs down—which is the negative z-direction—from a hook that has been nailed to

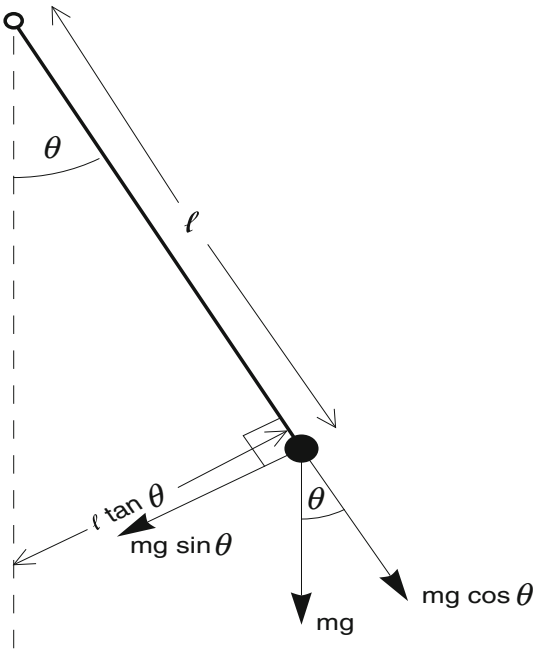


Fig. 8.5 Pendulum oscillates in two-dimensional motion in $x - z$ plane. When oscillating it experiences air resistance. At time t, it is seen moving past angle θ with the vertical

the ceiling. The pendulum is set to oscillate in two-dimensional motion in the $x - z$ plane. Air resistance can be approximated as a frictional force proportional to the speed with which the bob is moving—more precisely, α times its speed. The ensuing friction slows the oscillatory motion. Assume that at time t the bob—see Fig. 8.5—is moving past a position where the rod makes an angle θ_t with the vertical. Its equation of motion is

$$m\frac{\mathrm{d}^2\{l\ \tan\theta_t\}}{\mathrm{d}t^2} = -\alpha\frac{\mathrm{d}\{l\ \tan\theta_t\}}{\mathrm{d}t} - m\,g\sin\theta_t\ . \tag{8.41}$$

g is the acceleration due to gravity. For arbitrary size of the angle θ_t, (8.41) is nonlinear. However, when θ_t is much less than a radian both $\tan\theta_t$ and $\sin\theta_t$ tend to θ_t and the equation of motion becomes linear. [Note: π radians equals 180°. Thus, one radian is = 57.2958°.] We can then write (8.41) as

$$\frac{\mathrm{d}^2\theta_t}{\mathrm{d}t^2} + 2\mu\frac{\mathrm{d}\theta_t}{\mathrm{d}t} + \omega_0^2\,\theta_t = 0 \tag{8.42}$$

where

$$2\mu = \frac{\alpha}{m}\ ;\ \omega_0 = \sqrt{\frac{g}{l}}\ . \tag{8.43}$$

The equilibrium position (EP) refers to the case where the angle θ is zero: Meaning, the massless rod is exactly vertical.

Solution of (8.42)

Equation (8.42) is homogeneous linear ordinary differential equation (hlinODE). Therefore, its particular integral, I_{pi}, is vanishing. In order to determine its complementary solution, $S_{\text{comp}} \rightarrow \theta_t$, one needs first to solve its characteristic equation E_{ch}. To that end, $\frac{\mathrm{d}}{\mathrm{d}t}$ is replaced by k and $\frac{\mathrm{d}^2}{\mathrm{d}t^2}$ by k^2.

$$k^2 + 2\mu\, k + \omega_0^2 = 0 \quad . \tag{8.44}$$

The two solutions, $k \equiv k_1$ and k_2,

$$\begin{aligned} k_1 &= -\mu + \sqrt{(\mu^2 - \omega_0^2)} \ , \\ k_2 &= -\mu - \sqrt{(\mu^2 - \omega_0^2)} \ , \end{aligned} \tag{8.45}$$

of (8.44), lead to the complementary solution S_{comp}.

$$S_{\text{comp}} \equiv \theta_t \ = \ \sigma_1 \, \exp(k_1\, t) + \sigma_2 \, \exp(k_2\, t) \ . \tag{8.46}$$

For use later, it is convenient to also record its differentials.

$$\begin{aligned} \frac{\mathrm{d}\theta_t}{\mathrm{d}t} \ &\equiv \ \dot{\theta_t} \ = \ \sigma_1\, k_1 \, \exp(k_1\, t) + \sigma_2\, k_2 \, \exp(k_2\, t) \ ; \\ \frac{\mathrm{d}^2\theta_t}{\mathrm{d}t^2} \ &\equiv \ \ddot{\theta_t} \ = \ \sigma_1\, (k_1)^2 \, \exp(k_1\, t) + \sigma_2\, (k_2)^2 \, \exp(k_2\, t) \ . \end{aligned} \tag{8.47}$$

Because of convenience, and the fact that it does not at all affect any of the substance of this work, in what follows we shall work with $t \geq 0$.

8.2.1 Over-Damped Oscillating Pendulum

8.2.2 Angle θ_t

When $\mu^2 > \omega_0^2$, friction is overpowering. Then, the pendulum is said to be over-damped—see Figs. 8.6 and 8.7. Because μ is greater than zero, for an over-damped pendulum the following inequalities hold.

$$\mu > \sqrt{(\mu^2 - \omega_0^2)} \ ; \ \ k_1 < 0 \ ; \ \ k_2 < 0 \ ; \ \ k_1 > k_2 \ . \tag{8.48}$$

Therefore, irrespective of whether the constants σ_1 and/or σ_2 are positive or negative, according to (8.46) and (8.48) the angle θ_t must tend to the (EP) at long time. There is also a possibility that before reaching the equilibrium position (EP) the bob will move across to the other side, come to a momentary stop, reverse direction to head back, and eventually $\theta_t \to (EP)$ as $t \to \infty$. But much depends—see below—on the relative size of the two constants σ_1 and σ_2. [Note: Knowledge of two boundary conditions—for instance, the angle θ_0 and the velocity at a specified time—is needed to fix the two constants σ_1 and σ_2.]

At $t = 0$, the bob is at an angle θ_0 with respect to the vertical. According to (8.46)

$$\theta_0 = (\sigma_1 + \sigma_2) \quad . \tag{8.49}$$

For convenience, θ_0 is chosen to be positive. The direction of motion of the bob at $t = 0$ may, however, be positive or negative. Or indeed, the bob may be stationary at that moment.

8.2.3 Angle θ_t and Its Extrema

The following parameters have been set to be the same for all the four curves, A, B, C, and D, displayed in Fig. 8.6.

$$k_1 = -\frac{1}{3}\,;\; k_2 = -\frac{5}{3}\,;\; \left(\frac{k_2}{k_1}\right) = 5\,;\; \sigma_1 + \sigma_2 = 3\;;$$
$$\left(\frac{-k_1 - k_2}{2}\right) = \mu = 1\,;\; \left(\frac{k_1 - k_2}{2}\right) = \sqrt{(\mu^2 - \omega_0^2)} = \frac{2}{3}\;. \tag{8.50}$$

However, while the sum $\sigma_1 + \sigma_3 = 3$ is the same for all the four curves in Fig. 8.6, individual values of σ_1 and therefore σ_2 may differ from one curve to the other.

8.2.4 Bob Stays on Original Side Curves A–C

To understand how the angle θ_t changes with t, one needs to study its description in (8.46). Time moves forward. Given both k_1 and k_2 are negative, if σ_1 and σ_2 have the same sign, (8.46) keeps its sign and the bob never crosses over to the other side. Accordingly, for the curve marked A in Fig. 8.6, where $\sigma_1 = \sigma_2 = \frac{3}{2}$, the bob stays on the positive side.

Regarding curve B, it is convenient to rewrite (8.46) as

$$\theta_t = \sigma_1 \exp(k_1\, t)\left\{1 + \left(\frac{\sigma_2}{\sigma_1}\right)\exp[-(k_1 - k_2)\, t]\right\} \quad . \tag{8.51}$$

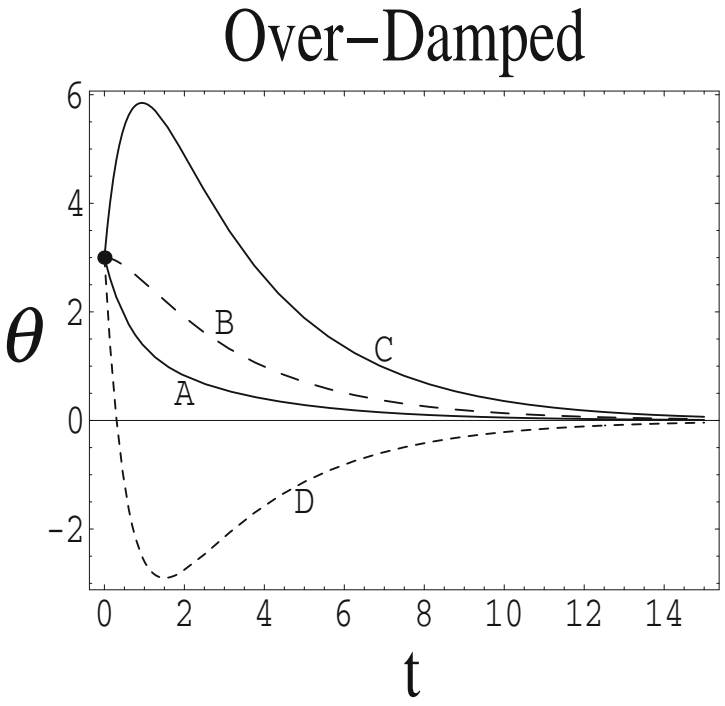


Fig. 8.6 Pendulum motion experiences friction strong enough that its motion is over-damped. The rod is turning past angle $\theta = 3°$ at time $t = 0$

Clearly, the term $\sigma_1 \exp(k_1\, t)$ does not change sign with passage of time. And if θ_t were to change sign, it will have to come from the second term, namely $\left\{1 + \left(\frac{\sigma_2}{\sigma_1}\right) \exp[-(k_1 - k_2)\, t]\right\}$. At $t = 0$, this term equals $1 + \left(\frac{\sigma_2}{\sigma_1}\right)$. And at $t \to \infty$, because $(k_1 - k_2) > 0$, it is equal to 1. These two values can have the same sign only if

$$\left(\frac{\sigma_2}{\sigma_1}\right) > -1 \quad . \tag{8.52}$$

The curve marked B in Fig. 8.6 refers to a pendulum that has $\sigma_1 = 3° \times \frac{5}{4}$, $\sigma_2 = -3° \times \frac{1}{4}$. Therefore, $\left(\frac{\sigma_2}{\sigma_1}\right) = -\frac{1}{5}$ is within the range specified by (8.52). Hence, the bob stays on the positive side. To determine whether the angle θ_t experiences an extremum, it is convenient to reprint (8.34) in terms of the present notation. It indicates the angle, and the angular velocity undergoes extremum at times

$$t_{\theta_{\text{t:extremum}}} = \left(\frac{1}{k_1 - k_2}\right) \log\left[-\frac{\sigma_2}{\sigma_1}\left(\frac{k_2}{k_1}\right)\right] \; ;$$
$$t_{\frac{\mathrm{d}\theta_t}{\mathrm{d}t}:\text{extremum}} = \left(\frac{1}{k_1 - k_2}\right) \log\left[-\frac{\sigma_2}{\sigma_1}\left(\frac{k_2}{k_1}\right)^2\right] \; . \tag{8.53}$$

Given $(k_1 - k_2) = \frac{4}{3}$, $\left(\frac{k_2}{k_1}\right) = 5$, and $\left(\frac{\sigma_2}{\sigma_1}\right) = -\frac{1}{5}$,

$$t_{\theta_{\text{t:extremum}}} \; = \; \left(\frac{3}{4}\right) \log\left[\frac{1}{5}\,(5)\right] \; = \; \left(\frac{3}{4}\right) \log\,(1) \; = \; 0 \; . \tag{8.54}$$

At time zero, according to (8.46) and (8.49), the angle is 3°. Its second derivative,

$$\left(\frac{\mathrm{d}^2\theta_t}{\mathrm{d}t^2}\right)_{t=0} \; = \; \sigma_1\,(k_1)^2 + \sigma_2\,(k_2)^2 \; = \; \left(\frac{15}{4}\right)\left(\frac{-1}{3}\right)^2 - \left(\frac{3}{4}\right)\left(\frac{-5}{3}\right)^2$$
$$= \; -\,4\sigma_1\,k_1^2 \; = \; -\,\frac{5}{3} \; , \tag{8.55}$$

is negative. Meaning, the bob starts its journey at $\theta_{t_{\theta_{\text{t:extremum}}}} \; = \; 3^\circ$, that is a maximum, and its distance from the (EP) continues decreasing monotonically with the passage of time.

For the curve C in Fig. 8.6, $\sigma_1 = 10$, $\sigma_2 = -7$ and the ratio $\frac{\sigma_2}{\sigma_1} = -0.7$ is again > -1. Thus, the bob stays on the positive side. Initially, at time $t_{\theta_{\text{t:extremum}}}$, the angle continues increasing and according to (8.53) reaches a maximum $\theta_{\text{t:extremum}}$, where it halts momentarily and starts decreasing.

$$t_{\theta_{\text{t:extremum}}} \; = \; \left(\frac{3}{4}\right) \log\,[0.7\,(5)] \; = \; 0.939572 \; .$$
$$\theta_{\text{t:extremum}} \; = \; \sigma_1 \exp\,(k_1\,t_{\theta_{\text{t:extremum}}}) \; + \; \sigma_2 \exp\,(k_2\,t_{\theta_{\text{t:extremum}}})$$
$$= \; 10 \exp\left[-\left(\frac{1}{3}\right) 0.9396\right] - 7 \exp\left[-\left(\frac{5}{3}\right) 0.9396\right] \; = \; 5.850^\circ \; .$$

Eventually—literally, at $t \to \infty$—the bob arrives (EP).

Curve D refers to the case $\sigma_1 = -6^\circ$, $\sigma_2 = 9^\circ$. As a result $\left(\frac{\sigma_2}{\sigma_1}\right) = -1.5 \not> -1$ and the bob crosses over to the other side before it stops at time, say, t_{min} at an angle, say, θ_{min}, reverses direction and moves toward the (EP) which is where its journey ends.

According to (8.53),

$$t_{\text{min}} \; \to = t_{\theta_{\text{t:extremum}}} \; = \; \left(\frac{3}{4}\right) \log\left[-\frac{\sigma_2}{\sigma_1} \times 5\right] \; = \left(\frac{3}{4}\right) \log(1.5 \times 5)$$
$$= 1.511 \; , \tag{8.56}$$

and (8.46),

$$\theta_{\min} = \sigma_1 \exp(k_1 t_{\min}) + \sigma_2 \exp(k_2 t_{\min}) \ ,$$
$$= -6 \exp\left(-\frac{1}{3} \times 1.511\right) + 9 \exp\left(-\frac{5}{3} \times 1.511\right) = -2.901^\circ. \quad (8.57)$$

8.2.5 *Extremum in $\theta_t{}^{\cdot}$*

The x axis in Fig. 8.7 is the time t, and the y axis is the angular velocity $\left(\frac{d\theta_t}{dt}\right) \equiv \theta_t{}^{\cdot}$ [See (8.47)].

Curves A, B, C, and D in Fig. 8.7 refer to the same parameters as those in Fig. 8.6. Because θ_t in curve A, Fig. 8.6, undergoes no extremum, curve A in Fig. 8.7 does not pass through $\theta_t{}^{\cdot} = 0$. On the other hand, because of the presence of an extremum

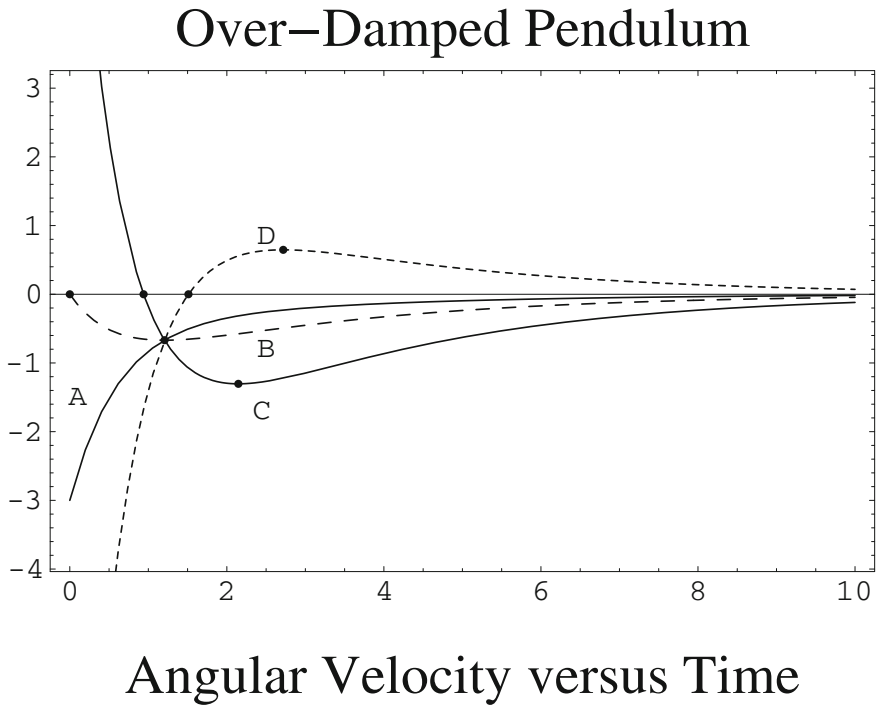


Fig. 8.7 x axis is time t and y axis is angular velocity $\theta_t{}^{\cdot}$. Curves A, B, C, and D pertain to the same values as corresponding curves in Fig. 8.6. Angular velocity passes through zero where angle θ_t reaches extremum

in θ_t in each of the other three curves in Fig. 8.6, the relevant curves pass through $\theta_t{}^{\cdot} = 0$. For curve B, this occurs at the beginning: that is, at $t = 0$. For curves C and D, $\theta_t{}^{\cdot} = 0$ occurs at 0.9396 and 1.511, respectively.

In addition to $\theta_t{}^{\cdot}$ itself, graphical treatment of extrema for $\theta_t{}^{\cdot}$ requires knowledge also of $\theta_t{}^{\cdot\cdot}$—these are both available in (8.47)—as well as that of $\theta_t{}^{\cdot\cdot\cdot}$, which is given below.

$$\begin{aligned}\theta_t{}^{\cdot\cdot\cdot} \equiv \frac{\mathrm{d}^3\theta_t}{\mathrm{d}t^3} &= \frac{\mathrm{d}^2\theta_t{}^{\cdot}}{\mathrm{d}t^2} = \sigma_1\,(k_1)^3\exp(k_1\,t) + \sigma_2\,(k_2)^3\exp(k_2\,t) \\ &= \sigma_1\,(k_1)^3\exp(k_1\,t)\left\{1 + \left(\frac{\sigma_2 k_2^3}{\sigma_1 k_1^3}\right)\exp[-(k_1 - k_2)\,t]\right\}. \qquad (8.58)\end{aligned}$$

Is there an extremum present in curve A, Fig. 8.7? To answer this query, use (8.53) and calculate the time $t = t_{\theta_{\mathrm{t:extremum}}}$ when such an extremum would occur. To that end recall that $(k_1 - k_2) = \frac{4}{3}$, and $\left(\frac{k_2}{k_1}\right) = 5$. Next set $\theta_t{}^{\cdot\cdot} = 0$ in (8.47) and rewrite the result as

$$t_{\theta_{\mathrm{t:extremum}}} = \left(\frac{1}{k_1 - k_2}\right)\log\left\{-\frac{\sigma_2\,k_2^2}{\sigma_1\,k_1^2}\right\} = \left(\frac{3}{4}\right)\log\left[-\left(\frac{\sigma_2}{\sigma_1}\right)25\right]. \qquad (8.59)$$

For curve A, $\left(\frac{\sigma_2}{\sigma_1}\right) = \frac{1.5}{1.5} = 1$. Therefore,

$$t_{\theta_{\mathrm{t:extremum}}} = \left(\frac{3}{4}\right)\log(-25)\ . \qquad (8.60)$$

This is a complex number. Because time $t_{\theta_{\mathrm{t:extremum}}}$ has to be real this result is unphysical. Thus, curve A does not undergo any extrema.

For curve B, $\left(\frac{\sigma_2}{\sigma_1}\right) = \frac{-3/4}{15/4} = -1/5$. Therefore, an extremum occurs at time

$$t_{\theta_{\mathrm{t:extremum}}} = \left(\frac{3}{4}\right)\log\left[-\left(\frac{-1}{5}\right)25\right] = \left(\frac{3}{4}\right)\log(5) = 1.20708. \qquad (8.61)$$

Using this value of time in (8.46), one can determine the relevant value of $\theta_t{}^{\cdot}$ at this extremum. The result is -0.668739. Because $\theta_t{}^{\cdot\cdot\cdot}$ at this time—namely 1.20708—

$$\begin{aligned}\theta_t{}^{\cdot\cdot\cdot} \equiv \frac{\mathrm{d}^3\theta}{\mathrm{d}t^3} &= \sigma_1\,(k_1)^3\exp(k_1\,t) + \sigma_2\,(k_2)^3\exp(k_2\,t) \\ &= 0.371521, \qquad (8.62)\end{aligned}$$

is positive, the extremum is a minimum.

For curve C, $\left(\frac{\sigma_2}{\sigma_1}\right) = \frac{-7}{10}$. Therefore, an extremum occurs also here. The relevant time is

$$t = \left(\frac{3}{4}\right) \log(5) = 2.14665. \tag{8.63}$$

Proceeding as before, the θ_t˙ for this extremum is -1.303795 and again it is a minimum.

For curve D, the extremum is a maximum and it occurs at $t = 2.18256$ and θ_t˙ $= 0.64565$.

8.2.6 *Critically Damped Pendulum*

When strength of friction becomes similar to the ordering tendency of the natural vibrations a pendulum is said to be critically damped—see Figs. 8.8 and 8.9. More precisely, this is the case when $\mu^2 = \omega_0^2$. Here, the characteristic equation (8.44) has two equal roots : $k_1 = k_2 = -\mu$. [See (8.45)] Therefore, according to the well-established procedure, and (3.37), the angle at time t can be expressed as

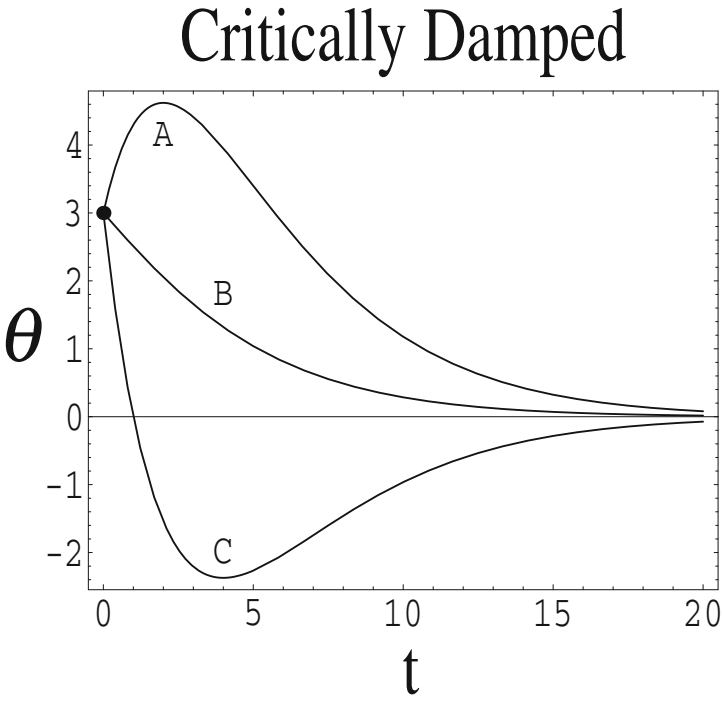


Fig. 8.8 Angle pendulum rod of critically damped pendulum makes it displayed as function of time. Curves A, B, and C are defined in (8.69)–(8.74)

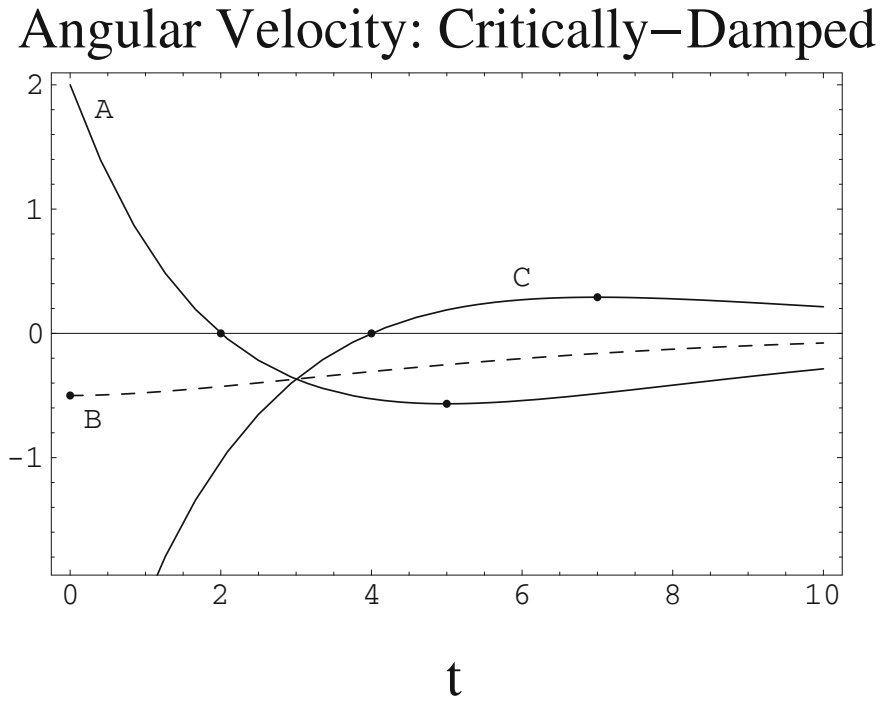


Fig. 8.9 Plotted along the vertical axis is the angular velocity, $\frac{d\theta}{dt}$, for a critically damped pendulum. Curves A, B, and C are the same as in Fig. 8.8

$$\theta_t = (\sigma_3 + \sigma_4\, t)\, \exp(-\mu\, t) \ . \tag{8.64}$$

Prediction from this equation is displayed in Fig. 8.8. At time $t = 0$, the bob is at a point where the rod, according to (8.64), makes an angle $\theta_0 = \sigma_3$ (which was set equal to $+3°$). Because $\mu > 0$, as time $t \to \infty$, the angle θ_t tends to (EP). In between, there are two possibilities: First, the angle stays positive, eventually reaching (EP). And the second, the angle switches over to the negative side where at time, say t_{critical}, the angle reaches, say $\theta_{t_{\text{critical}}}$, where it comes to a momentary stop. Instantly, the bob starts the reverse journey, eventually reaching the end of its travel at the midpoint: that is, when the angle θ_t reaches the (EP).

Let us examine this behavior and see exactly what transpires. To this end, in Fig. 8.9, look at the derivative of θ_t.

$$\frac{d\theta_t}{dt} = (\sigma_4 - \mu\sigma_3 - \mu\, \sigma_4\, t)\, \exp(-\mu\, t) \ . \tag{8.65}$$

Because $\frac{d\theta_t}{dt}$ must tend to zero at $t = t_{\text{critical}}$, $(\sigma_4 - \mu\sigma_3 - \mu\,\sigma_4\,t_{\text{critical}}) = 0$. As such

$$t_{\text{critical}} = \frac{\sigma_4 - \mu\sigma_3}{\mu\,\sigma_4} \quad = \quad \left(\frac{1}{\mu} - \frac{\sigma_3}{\sigma_4}\right) \quad . \tag{8.66}$$

Therefore, according to (8.64) and (8.66),

$$\begin{aligned} \theta_{t_{\text{critical}}} &= (\sigma_3 + \sigma_4\,t_{\text{critical}})\exp(-\mu\,t_{\text{critical}}) \\ &= \left(\frac{\sigma_4}{\mu}\right)\exp\left(-\mu\,t_{\text{critical}}\right) \quad . \end{aligned} \tag{8.67}$$

In order to determine the type of extremum that $\theta_{t_{\text{critical}}}$ represents, one needs to look at the second derivative $\frac{d^2\theta_t}{dt^2}$ at time $t = t_{\text{critical}}$. One knows that friction is positive therefore the parameter $\mu > 0$. And if the parameter σ_4 should also be > 0, the second differential of equation (8.65) would be clearly negative.

$$\left(\frac{d^2\theta_t}{dt^2}\right)_{t=t_{\text{critical}}} = \exp\left\{-\mu\left(\frac{1}{\mu} - \frac{\sigma_3}{\sigma_4}\right)\right\}(-\mu\,\sigma_4) < 0, \tag{8.68}$$

As such, $\theta_{t_{\text{critical}}}$ would be a maximum. At this juncture, it is helpful to give $\theta_{t_{\text{critical}}}$ a more informative new name: call it $\theta_{\max}$. For the same reason, also re-name the relevant t_{critical} as $t_{\max}$. And rewrite (8.66) and (8.67) as

$$t_{\max} = \frac{\sigma_4 - \mu\sigma_3}{\mu\,\sigma_4} \quad = \quad \left(\frac{1}{\mu} - \frac{\sigma_3}{\sigma_4}\right) \quad . \tag{8.69}$$

and

$$\theta_{\max} = \left(\frac{\sigma_4}{\mu}\right)\exp\left(-\mu\,t_{\max}\right) \quad . \tag{8.70}$$

Such behavior is demonstrated in curve A in Fig. 8.8. For this curve, $\sigma_3 = \sigma_4 = 3°$ and $\frac{1}{\mu} = 3$. Therefore

$$t_{\max} = \left(\frac{1}{\mu} - \frac{\sigma_3}{\sigma_4}\right) \quad = \quad 2 \quad , \tag{8.71}$$

and

$$\theta_{\max} = \left(\frac{\sigma_4}{\mu}\right)\exp\left(-\mu\,t_{\max}\right) = 3° \times 3 \times \exp\left(-\frac{2}{3}\right) = 4.621° \quad . \tag{8.72}$$

Next, in Fig. 8.8, examine curve B where $\sigma_3 = 3°$, $\frac{1}{\mu} = 3$, and $\sigma_4 = 0.5°$. But now, because the predicted $t_{\max}$ is equal to -3, which is a time before the experiment began, the requirement for an observed maximum is not satisfied. As a result, while the angle stays positive, it continues to decrease monotonically heading to the (EP) as t increases toward ∞.

The third curve, C in Fig. 8.8, refers to the case where $\frac{1}{\mu} = 3$, $\sigma_4 = -\sigma_3 = -3$. Thus, $\mu\,\sigma_4$ is < 0. With reference to (8.68), $(-\mu\,\sigma_4)$, and therefore $\left(\frac{\mathrm{d}^2\theta_{t_{\text{critical}}}}{\mathrm{d}t^2}\right)_{t=t_{\text{critical}}}$ are positive. As a result, the relevant θ_{critical} is a minimum. Again, it is helpful to call this value of θ_{critical} by a new name: θ_{min}, and the time it is reached as $t = t_{\text{min}}$. Thus

$$t_{\text{min}} \;=\; \left(\frac{1}{\mu} - \frac{\sigma_3}{\sigma_4}\right) \;=\; 3 - \left(\frac{3}{-3}\right) \;=\; 4 \quad , \tag{8.73}$$

and

$$\theta_{\text{min}} = -\left(\frac{|\sigma_4|}{\mu}\right) \exp\left(-\mu\, t_{\text{min}}\right) = -\left(3^\circ \times 3\right) \exp\left(-\frac{4}{3}\right) = -2.372^\circ \tag{8.74}$$

With the progress of time, due to the presence of negative exponent in (8.64), $\theta_t \rightarrow (EP)$.

8.2.7 *Under-Damped Motion*

When friction is weaker than the critical amount, i.e., $\omega_0^2 > \mu^2$, the pendulum is under-damped—see Fig. 8.10. The two roots, k_1 and k_2—compare (8.45)—of the characteristic equation (8.44) are complex, namely

$$k_1 \;=\; -\mu - i\sqrt{\omega_0^2 - \mu^2}\;;\; k_2 \;=\; -\mu + i\sqrt{\omega_0^2 - \mu^2} \quad . \tag{8.75}$$

Using a procedure similar to that in (3.21)–(3.24), the complementary solution is readily found.

$$\begin{aligned} S_{\text{comp}} \rightarrow \theta_{\text{under-damped}} \;&=\; \sigma_1 \exp\left(k_1\, t\right) \;+\; \sigma_2 \exp\left(k_2\, t\right) \;, \\ =\; \exp\left(-\mu\, t\right) &\left[\sigma_3 \sin\left(t\sqrt{\omega_0^2 - \mu^2}\right) + \sigma_4 \cos\left(t\sqrt{\omega_0^2 - \mu^2}\right)\right] \;, \\ &=\; \sigma_5 \exp\left(-\mu\, t\right) \cos\left(t\sqrt{\omega_0^2 - \mu^2} - \phi\right) \;, \end{aligned} \tag{8.76}$$

where $\sigma_3 = -i(\sigma_1 - \sigma_2)$, $\sigma_4 = (\sigma_1 + \sigma_2)$, and $\sigma_5 = \left(\sqrt{\sigma_3^2 + \sigma_4^2}\right)$. Also, $\sin\phi = \left(\frac{\sigma_3}{\sigma_5}\right)$ and $\cos\phi \;=\; \left(\frac{\sigma_4}{\sigma_5}\right)$. Because $\mu > 0$, as $t \rightarrow \infty$ the angle $\theta_{\text{under-damped}}$ tends to (EP). However, before the bob comes to the final absolute stop, it repeats the following performance many times—in principle, infinite number of times.

At time $t = 0$ and angle 3°, consider the bob is heading left. As it moves toward decreasing angle, it crosses the (EP), proceeds further left, slows down for a momen-

tary halt, reverses its direction, and starts moving forward to the right: crosses the (EP) again before coming to a momentary stop on the right side and starting another journey leftward. This behavior contrasts with that of the over-damped and critically damped pendulums that either never move across the midpoint or cross it at most only one time.

Another point to note is that according to (8.76), the angle $\theta_{\text{under-damped}}$ reaches a maximum $= \theta_n$ at a time $t = t_n$ if $\left(t_n\sqrt{\omega_0^2 - \mu^2} - \phi\right) = 2n\pi$ and the next maximum $= \theta_{n+1}$ at time t_{n+1} when $\left(t_{n+1}\sqrt{\omega_0^2 - \mu^2} - \phi\right) = 2(n+1)\pi$. Because the cosines at both these times are equal to unity, the ratio of the angles at these two successive maxima is

$$\frac{\theta_n}{\theta_{n+1}} = \frac{\exp(-\mu\, t_n)}{\exp(-\mu\, t_{n+1})} = \exp\left(\frac{2\pi\mu}{\sqrt{\omega_0^2 - \mu^2}}\right). \tag{8.77}$$

[Note : The same is also true for the ratio of angles at two successive minima.] Remarkably, this ratio is not dependent on n. Rather, it is the same for any two successive maxima (or even, any two successive minima). Also, it is readily measurable. Another quantity that is easy to measure is the cycle time, δ_{cycle}, of the damped oscillation. That is

$$\delta_{\text{cycle}} = t_{n+1} - t_n = \left(\frac{2\pi}{\sqrt{\omega_0^2 - \mu^2}}\right). \tag{8.78}$$

From δ_{cycle} and the ratio $\log\left(\frac{\theta_n}{\theta_{n+1}}\right)$, one can determine the strength of friction,

$$\mu = \left(\frac{\log\left(\frac{\theta_n}{\theta_{n+1}}\right)}{\delta_{\text{cycle}}}\right), \tag{8.79}$$

which is otherwise hard to measure.

For an under-damped pendulum, a typical plot of the angle versus time could be that given in Fig. 8.8. Here, ϕ has been set $= 0$, and σ_5 set at 3°. Also, $\frac{1}{\mu}$ is chosen to be 1.5 and $\sqrt{\omega_0^2 - \mu^2}$ is set at $\frac{10°}{\text{time}}$. In other words, Fig. 8.8 refers to

$$\begin{aligned}\theta_{\text{ud:transient}} &= \sigma_5 \exp(-\mu\, t)\cos\left(t\sqrt{\omega_0^2 - \mu^2}\right) \\ &= 3\exp\left(-\frac{t}{1.5}\right)\cos(10\, t)\ .\end{aligned} \tag{8.80}$$

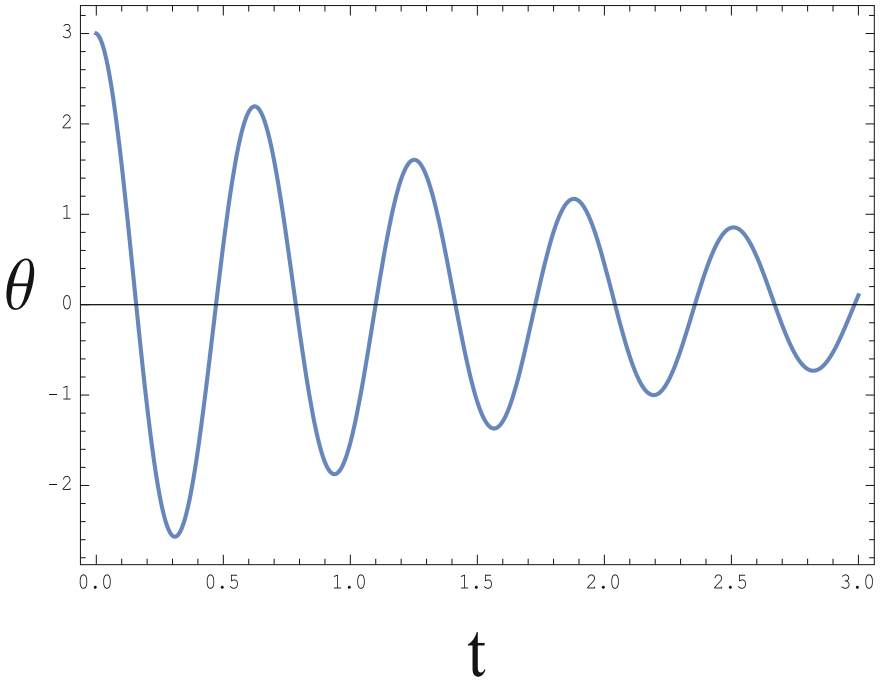


Fig. 8.10 With sinusoidal external force, angle θ the pendulum makes with the vertical is displayed as a function of time. Plotted here is (8.80)

Therefore, according to (8.77), in Fig. 8.10 the ratio of any two successive maxima, or minima, for the angle $\theta_{\text{ud:transient}}$ is equal to $\exp\left(\frac{2\pi}{1.5\times 10}\right) = 1.520$.

8.2.8 Steady-State Motion

Transient state motion of damped pendulums was analyzed in the foregoing. The steady-state motion, caused by the application of an applied force, is considered below.

In the presence of an applied force, say $m\,l\,f(t)$, the original equation of motion, namely (8.41), changes to the following:

$$m\frac{\mathrm{d}^2\{l\,\tan\theta\}}{\mathrm{d}t^2} + \alpha\frac{\mathrm{d}\{l\,\tan\theta\}}{\mathrm{d}t} + m\,g\sin\theta \;=\; m\,l\,f(t) \quad . \tag{8.81}$$

For small angle θ, it can be expressed as

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + 2\mu\frac{\mathrm{d}\theta}{\mathrm{d}t} + \omega_0^2\,\theta \;=\; f(t) \tag{8.82}$$

where $\omega_0 = \sqrt{\frac{g}{l}}$ and μ is $= \frac{\alpha}{2m}$.

8.2.9 *Sinusoidal External Force: Steady State*

Oscillating Pendulum

Assume the externally applied force is sinusoidal in time with angular velocity ω.

$$m\,l\,f(t) \;=\; m\,l\,A\,\cos(\omega\,t)\;\;;\;\;\text{i.e.}\;\;;\;\;f(t) \;=\; A\,\cos(\omega\,t)\;\;. \tag{8.83}$$

With the application of external force (8.83), the equation of motion (8.82) becomes inhomogeneous linear (ODE) with constant coefficients. The external force, however, does not affect the complementary solution that relates to the transient states discussed in detail in the foregoing. Therefore, only the particular integral, I_{pi}, that leads to steady-state motion needs to be evaluated here. Denoting $\Delta = \frac{\mathrm{d}}{\mathrm{d}t}$, the particular integral of equation (8.82) is calculated in the usual fashion as follows.

$$\begin{aligned}
\theta_{pi} \to I_{pi} \;&=\; A\frac{1}{\Delta^2 + 2\mu\Delta + \omega_0^2}[\cos(\omega\,t)] \\
&=\; A\frac{1/2}{\Delta^2 + 2\mu\Delta + \omega_0^2}[\exp(i\,\omega\,t) + \exp(-i\,\omega\,t)] \\
&=\; A\exp(i\,\omega\,t)\frac{1/2}{-\omega^2 + 2\mu\,i\,\omega + \omega_0^2} + \exp(-i\,\omega\,t)\frac{1/2}{-\omega^2 - 2\mu\,i\,\omega + \omega_0^2} \\
&=\; A\frac{(\omega_0^2 - \omega^2)\cos(\omega t) + 2\mu\omega\sin(\omega t)}{(\omega_0^2 - \omega^2)^2 + 4\mu^2\omega^2} \\
&=\; \frac{A\cos(\omega t - \Phi)}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\mu^2\omega^2}} \;=\; \frac{\left(\frac{A}{\omega_0^2}\right)\cos(\omega t - \Phi)}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_0}\right)^2\right]^2 + \left(\frac{2\mu}{\omega_0}\right)^2\left(\frac{\omega}{\omega_0}\right)^2}} \\
&=\; M_{\text{ratio}} \times \theta_{\text{static deflection}} \times \cos(\omega t - \Phi).
\end{aligned} \tag{8.84}$$

The following expression in (8.84) is the magnification ratio M_{ratio}.

$$\begin{aligned} M_{\text{ratio}} &= \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right]^2+\left(\frac{2\mu}{\omega_0}\right)^2\left(\frac{\omega}{\omega_0}\right)^2}} \\ &= \frac{1}{\sqrt{1+\left(\frac{\omega}{\omega_0}\right)^4-\left(\frac{\omega}{\omega_0}\right)^2\left[2-\left(\frac{2\mu}{\omega_0}\right)^2\right]}} , \end{aligned} \tag{8.85}$$

The magnification ratio is unity when the applied force is constant in time: that is, $M_{\text{ratio}} = 1$ when $\omega = 0$. And when the frequency of the applied force reaches the undamped natural frequency of the pendulum, that is when $\omega \to \omega_0$, $M_{\text{ratio}} \to \left(\frac{\omega_0}{2\mu}\right)$. Next, in (8.84), is the quantity $\left(\frac{A}{\omega_0^2}\right)$ which is the static deflection, $\theta_{\text{static deflection}}$. This nomenclature is owed to the fact that in the absence of time-dependent motion the applied force would deflect the equilibrium position of the bob through an angle $\approx \frac{m\,\ell\,A}{m\,g} = \frac{A}{\omega_0^2}$. Also, in (8.84), Φ is the so-called lag angle defined by the relations

$$\begin{aligned} \cos(\Phi) &= \frac{(\omega_0^2-\omega^2)}{\sqrt{(\omega_0^2-\omega^2)^2+4\mu^2\omega^2}} , \\ \sin(\Phi) &= \frac{2\mu\omega}{\sqrt{(\omega_0^2-\omega^2)^2+4\mu^2\omega^2}} . \end{aligned} \tag{8.86}$$

For convenience, let us introduce the notation

$$\begin{aligned} x &= \left(\frac{\omega}{\omega_0}\right)^2 \quad ; \quad \sigma_c = \left[2-\left(\frac{2\mu}{\omega_0}\right)^2\right] \quad ; \\ z &= 1+x^2-x\,\sigma_c \ . \end{aligned} \tag{8.87}$$

Given the second derivative of z is a positive constant for all x, namely $\left(\frac{\mathrm{d}^2 z}{\mathrm{d}x^2}\right) = 2$, find the location where the first derivative of z is equal to zero. That is

$$\begin{aligned} \left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)_{x=x_c} &= 0 \\ &= (2x_c-\sigma_c) \ . \end{aligned} \tag{8.88}$$

This defines a location x_c,

$$x_c = \frac{\sigma_c}{2} = \frac{1}{2}\left[2 - \left(\frac{2\mu}{\omega_0}\right)^2\right] = 1 - 2\left(\frac{\mu}{\omega_0}\right)^2 \quad , \tag{8.89}$$

where z reaches a minimum, and as a result, M_{ratio} reaches a maximum.

$$(M_{\text{ratio}})_{\max} = \frac{1}{\sqrt{z}} = \frac{1}{\sqrt{1 - \frac{{\sigma_c}^2}{4}}} = \frac{1}{\sqrt{1 - x_c^2}} \quad . \tag{8.90}$$

Note when friction decreases, x_c increases toward unity. As a result, $1 - x_c^2$ decreases and the magnification ratio increases. Indeed, when there is no friction at all present, meaning when $\mu = 0$, $(M_{\text{ratio}})_{\max} \to \infty$.

Chapter 9
Resistors, Inductors, Capacitors



Introduced first are Kirchhoff's two rules that state: 'The incoming current at any given point equals the outgoing current at that point,' and 'The algebraic sum of changes in potential encountered by charges traveling, in whatever manner, through a closed-loop circuit is zero.' Included next is the Ohm's law: 'In a closed-loop circuit that contains a battery operating at V volt, and a resistor of strength R ohms, current flow is I amperes: $I = \frac{V}{R}$.' Several problems relating to additions of finite numbers of resistors, placed in various configurations, some in series and some in parallel formats, are worked out—see (9.2)–(9.30) and Figs. 9.1, 9.2, 9.3, 9.4, 9.5 and 9.6. More involved problems relating to total resistance and current flows in infinite networks of resistors are treated next—see (9.31)–(9.60) and Figs. 9.7, 9.8, 9.9, 9.10, 9.11, 9.12, 9.13, 9.14, 9.15, 9.16, 9.17. Electric circuit elements are listed in (9.61). Inductors are introduced in (9.62). Series and parallel circuits constituted of finite number of inductors, and infinite series–parallel circuits of inductors are treated in (9.63)–(9.70) and displayed in Figs. 9.18 and 9.19. The same is done also for capacitors—see (9.71)–(9.82) and Figs. 9.20 and 9.21. Resistor–capacitor circuits are treated in (9.83) to (9.86), and some results are displayed in Fig. 9.22. Resistor–inductor circuits are studied in (9.91)–(9.96), and results are plotted in Fig. 9.23. Inductor–capacitor circuits are analyzed in (9.97)–(9.106), and results are displayed in Fig. 9.23.

9.1 Electric Current

9.1.1 Kirchhoff's Rules

Central to the understanding of current flow are a few rules that are obeyed by the current and the circuitry through which it flows.

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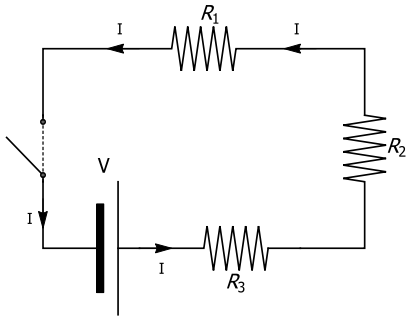


Fig. 9.1 With the switch on, the battery operates at constant voltage V and drives current I amps through resistors $R_3,\ R_2,\ R_1$ that are connected in series

(b)

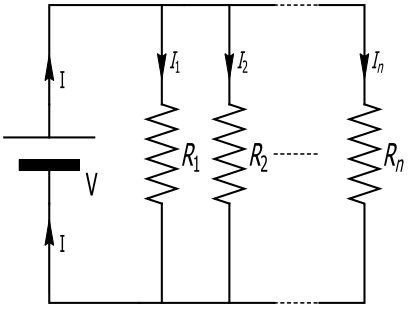


Fig. 9.2 Battery operating at voltage V drives currents $I_1,\ I_2, \ldots,\ I_n$ amps through resistors $R_1,\ R_2, \ldots\ R_n$ connected in parallel

(c)

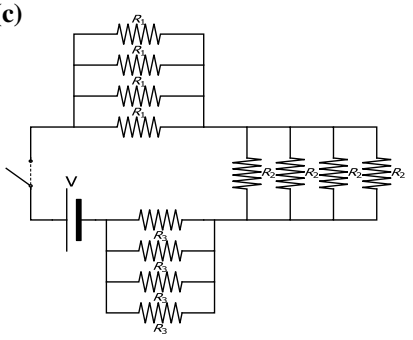


Fig. 9.3 Shown is the dd, the duodectet, that in addition to twelve resistors also has a switch and a battery

(d)

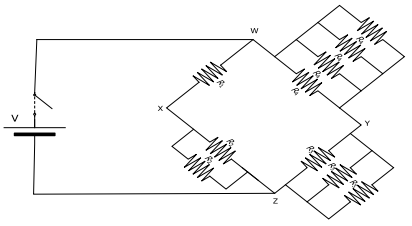


Fig. 9.4 Shown is circuit, an on–off switch, a battery operating at V volts, and variously put together four resistors: $R_1,\ R_2,\ R_3$ and R_4

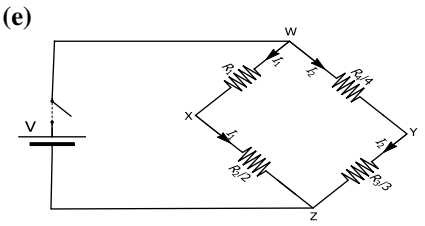


Fig. 9.5 This replicates the circuit in Fig. 9.4. With the switch on, the battery drives current I_1 through resistors $(R_1 + \frac{R_2}{2})$ and I_2 through $(\frac{R_4}{4} + \frac{R_3}{3})$

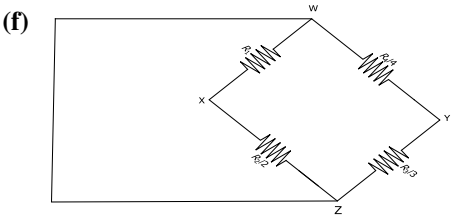


Fig. 9.6 Points W and Z are a single point WZ. Effective resistances between X and WZ and between WZ and Y are $R_{totalX-WZ}$ and $R_{totalWZ-Y}$

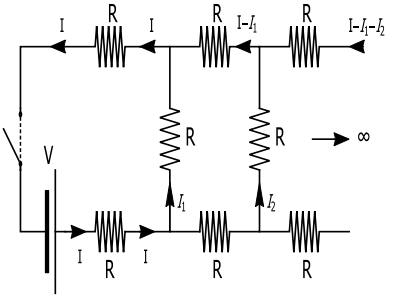


Fig. 9.7 For convenience, current is drawn as I

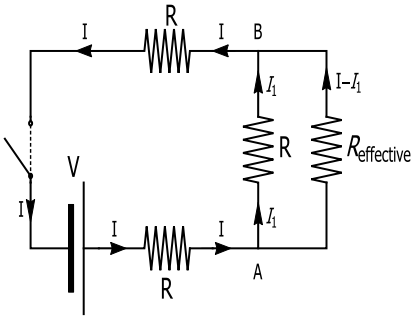


Fig. 9.8 For convenience, current is drawn as I

Kirchhoff's First Rule

(a): Electric charges, like billiard balls, flow down from points at higher electric potential to points at lower potential.

(b): When positive electric charge is placed at a given point, electric potential at the point rises. And the charge stays at the point only if the neighborhood is at even higher potential or if the charge is placed in a capacitor that is designed for the task

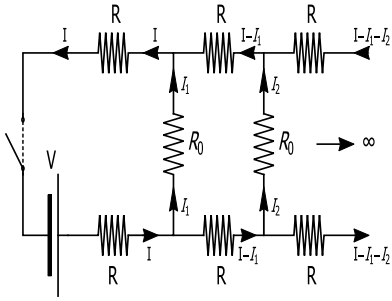


Fig. 9.9 For convenience, current is drawn as I. With the switch on, the battery drives current I through the infinite network shown. At the end of its journey, current I returns

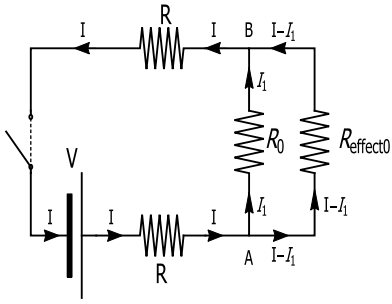


Fig. 9.10 For convenience, current is drawn as I. Figure 9.10 circuit represents the essentials of Fig. 9.9. The resistance $R_{effect0}$ equals the resistance of the infinite network that would remain after the first-trio, R, R_0, R has been subtracted

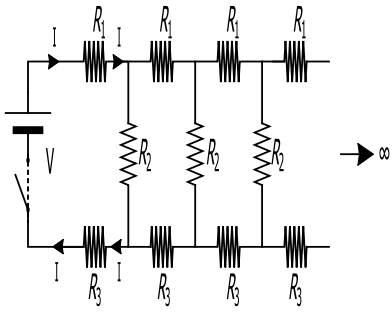


Fig. 9.11 For convenience, the current is drawn as I. With the switch on, the battery operating at voltage V drives current I through the infinite array of resistors shown

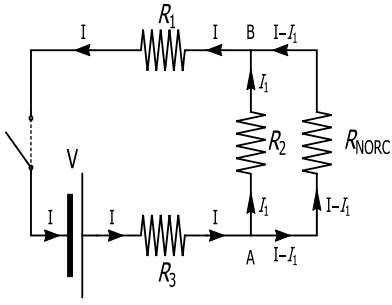


Fig. 9.12 For convenience, the current is drawn as I. With the switch on, the battery drives current I through the infinite array of resistors shown in Fig. 9.11. The effective resistance of the infinite circuit is R_{NORC}

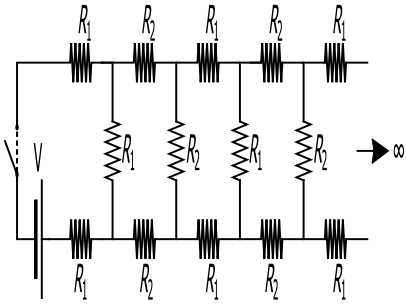


Fig. 9.13 With the switch on, the battery drives current through an infinite number of resistors R_1 and R_2 arrayed in the manner displayed

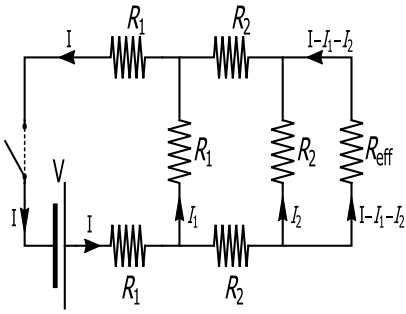


Fig. 9.14 For convenience, the current is drawn as I. In the 'septet' the current I splits into three parts: I_1, I_2, and $I - I_1 - I_2$

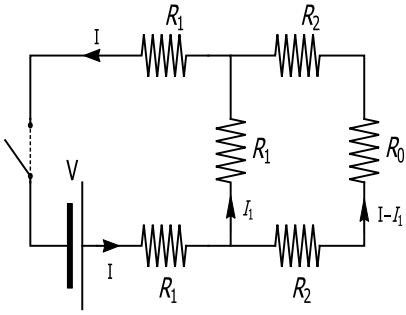


Fig. 9.15 For convenience, the current is drawn as I. In the 'equivalent-sextet,' the battery, operating at voltage V, drives current I that splits into I_1 and $I - I_1$

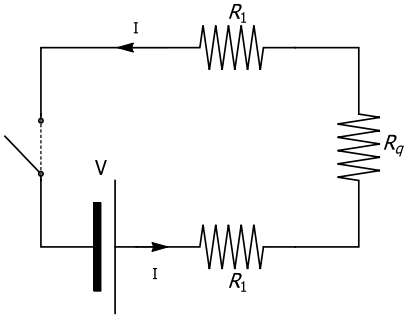


Fig. 9.16 For convenience, current is drawn as I. In the 'equivalent-triplet,' the battery drives current I through resistors R_1, R_q and R_1

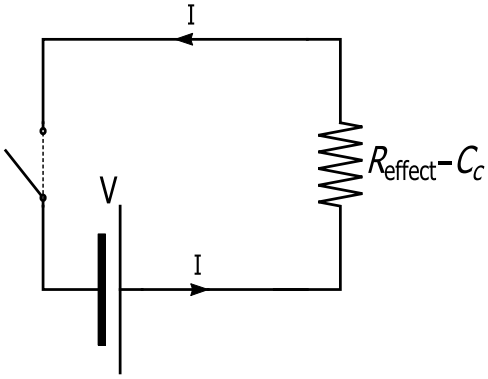


Fig. 9.17 For convenience, current is drawn as I. In the 'equivalent-singlet,' the battery drives current I through resistor $R_{effect} - Cc$ which is the effective resistance of the infinite circuit shown in Fig. 9.13

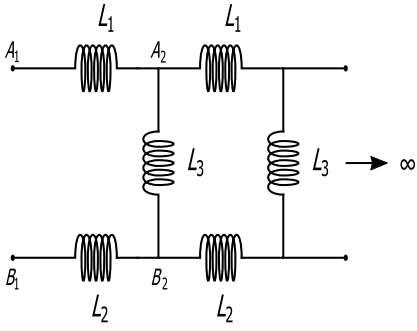


Fig. 9.18 An infinite array of three-upon-three inductors, L_1, L_2, L_3, is put together in the manner shown

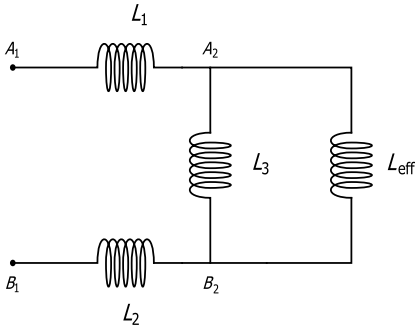


Fig. 9.19 The effective inductance of the infinite array of inductors shown in the Fig. 9.18 is equivalently represented by inductance of a single inductor L_{eff}

of holding charge. Lacking the capacitor, electric charge does not accumulate at a point. As such, at any point, the inflow of charge during a given time interval equals the outflow of charge during the same time interval.

The rate of flow of charge, $(q + \delta q - q)$, during a time interval $(t + \delta t - t)$—that is, $\frac{\delta q}{\delta t}$—is called electric current at the given time t. Thus, except for the extraordinary circumstance referred to above, incoming current at any point equals the outgoing current at that point. This is called Kirchhoff's first rule.

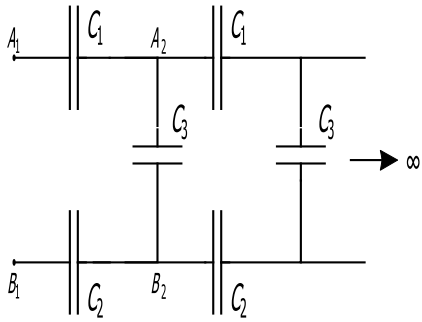


Fig. 9.20 An infinite array of three-upon-three capacitors, C_1, C_2, C_3, is put together in the manner displayed

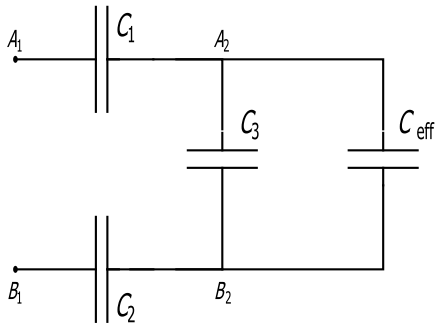


Fig. 9.21 Effective capacitance of the three-upon-three capacitors C_1, C_2, C_3, drawn in Fig. 9.20 is C_{eff}

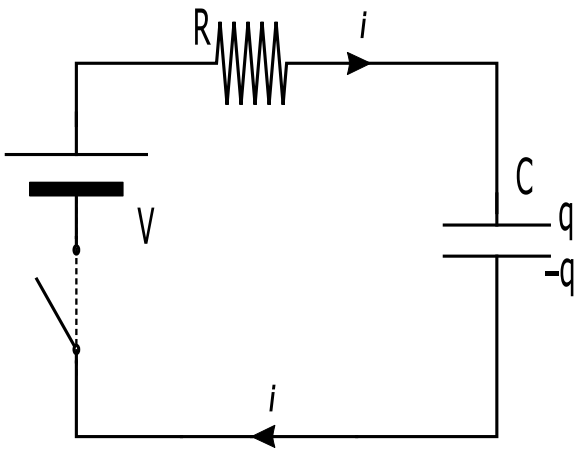


Fig. 9.22 Displayed is a circuit connected in series to an on–off switch, a battery, a resistor R, and a capacitor C. At time t the current is $i(t)$ and the charges on the plates are q(t) and −q(t)

Kirchhoff's Second Rule

Consider a person who starts off walking at a given point and engages in closed-loop travel: Meaning, he walks for however long, in whatever manner, up and down and sideways, but arranges to return exactly to the same point where he started. Adding diligently the changes in his height during the closed-loop travel, the sum must surely be zero because the person ends up exactly at the same height as he started.

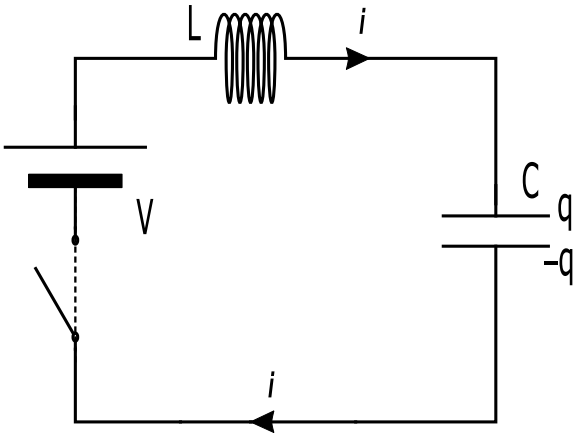


Fig. 9.23 Displayed is a circuit with a battery that at time t supplies current $i(t)$ to a series connection constituted of an inductor L, a capacitor C, and an on–off switch

Kirchhoff's second rule asserts that much like this person, if a given charge travels in a closed-loop circuit in whatever manner, the algebraic sum of changes in potential encountered must be zero.

9.1.2 Ohm's Law

According to Ohm's law, current I amperes flows through a closed-loop circuit that consists of a battery operating at voltage V volts and a resistor of strength R ohms.

$$I\ amperes = \frac{V\ volts}{R\ ohms}. \tag{9.1}$$

9.1.3 Addition of Resistors

Three Resistors in Series

With the switch on, the large battery, shown in Fig. 9.1, operates at V volts and drives current I amps through three resistors of strength R_1, R_2, and R_3 ohms that are connected in series, thereby forming a closed-loop series circuit. The physical result of the current flow through the circuit is found not to depend upon which of the three resistors leads, which is in the middle, or which follows. To understand the behavior of this series circuitry, consider a person trekking through a tunnel of length $d1$ and immediately thereafter through another similar tunnel of length $d2$ and yet another third similar tunnel of length $d3$. Clearly, it would take the person same amount of effort as trekking through a similar single tunnel of length $(d1 + d2 + d3)$. Accordingly, the effective resistance of three resistors, R_3, R_2, and

R_1 ohms placed one after the other—that is, as shown in Fig. 9.1, in series—is $R_{series-3}(effective) = (R_3 + R_2 + R_1)$ ohms.

n Resistors in Series

By an argument similar to that given above, the effective resistance of n resistors $R_1, R_2, \ldots, R_n$ connected in series should be $R_1 + R_2 + \cdots + R_n$. However, more formally, we can argue as follows. According to ohm's $law^{23.}$, when a current I $amperes^{31.}$ flows through a resistor of strength R_1 ohms the potential across it decreases $I\,R_1$ $volts^{32.}$. When current I flows through n resistors in series, the potential decrease across R_1 is $I\,R_1$, that across R_2 is $I\,R_2, \ldots$, and so on. Therefore, the total decrease in potential across the n resistors is $I\,(R_1 + R_2 + \cdots + R_n)$: exactly the same as it would be across a single resistor $R_{series-n}(effective)$.

$$R_{series-n}(effective) \;=\; R_1 + R_2 + \cdots + R_n \;. \tag{9.2}$$

For large number of resistors in series, the effective total resistance is large.

2 Resistors in Parallel

Consider two resistors R_1 and R_2 joined together in parallel and connected to a battery supplying direct current at constant voltage V volts. The parallel connection implies that while the positive outlet of the battery is connected to one end of each of the two resistors, the negative outlet is connected to the other end of the same two resistors. As a result, the potential difference across each of the two resistors is identical: equal to V volts. Assume this process results in current I_1 across resistor R_1 and I_2 across R_2. Then according to Ohm's law

$$\begin{aligned} I_1 &= \frac{V}{R_1} \\ I_2 &= \frac{V}{R_2} \;. \end{aligned} \tag{9.3}$$

The total current, $I_{parllel-2}(effective)$, that flows through the two resistors R_1 and R_2 that are connected in parallel, and their total resistance, $R_{parallel-2}(effective)$, are

$$\begin{aligned} I_{parllel-2}(effective) = I_1 + I_2 \;&=\; \left(\frac{V}{R_1} + \frac{V}{R_2}\right) \\ &\equiv \frac{V}{R_{parallel-2}(effective)}. \end{aligned} \tag{9.4}$$

Eliminating V from (9.4) leads to the relationship

$$\frac{1}{R_{parallel-2}(effective)} \;=\; \frac{1}{R_1} + \frac{1}{R_2}. \tag{9.5}$$

The resistance of two resistors R_1 and R_2 connected in parallel, namely

$$R_{parallel-2}(effective) \;=\; R_1 \parallel R_2 \equiv \frac{R_1.\,R_2}{R_1 + R_2} \tag{9.6}$$

has some interesting features. [Note: The notation $a \parallel b$ stands for $\frac{a.b}{a+b}$.] To examine these features, keep R_1 constant and make changes in R_2.

Process (1) : Consider the case $\frac{R_2}{R_1} \ll 1$. To this end, re-write (9.6) as

$$R_{parallel-2}(effective) \;=\; \frac{R_2}{1 + \left(\frac{R_2}{R_1}\right)} \tag{9.7}$$

and expand it in powers of the small parameter $\left(\frac{R_2}{R_1}\right)$.

$$R_{parallel-2}(effective) \;=\; R_2 \left\{1 - \left(\frac{R_2}{R_1}\right) + O\left(\frac{R_2}{R_1}\right)^2\right\}. \tag{9.8}$$

This result is interesting. When two resistors are connected in parallel, and one of the resistors is much smaller than the other one, the effective resistance is even smaller than the smaller resistor !

Process (2) : Increase R_2 till $R_2 = R_1$. Even though there are two resistors of strength R_1, their total resistance is only half of R_1.

$$R_{parallel-2}(effective) \;=\; \left(\frac{R_1}{2}\right)_{\lim R_2 \to R_1}. \tag{9.9}$$

Process (3) : Not surprisingly, an increase in R_2 beyond R_1 increases $R_{parallel-2}(effective)$ beyond $\left(\frac{R_1}{2}\right)$. But, the rate of increase is even less than a quarter of what one might expect. For instance, for a small increase $\left(\frac{R_2 - R_1}{R_2}\right) \ll 1$:

$$R_{parallel-2}(effective) = R_1 \parallel R_2 = \left(\frac{R_1}{2}\right)\left[\frac{1}{1 - \left(\frac{R_2 - R_1}{2\,R_2}\right)}\right]$$

$$= R_1\left[\frac{1}{2} + \left(\frac{R_2 - R_1}{4\,R_2}\right) - O\left(\frac{R_2 - R_1}{2\,R_2}\right)^2\right]. \tag{9.10}$$

Process (4) : With further increase in R_2, the effective resistance, albeit slowly, rises further. When $R_2 \gg R_1$, (9.6) gives

$$R_{parallel-2}(effective) = \frac{R_1}{1 + \left(\frac{R_1}{R_2}\right)} = R_1\left\{1 - \left(\frac{R_1}{R_2}\right) + O\left(\frac{R_1}{R_2}\right)^2\right\}. \tag{9.11}$$

The effective resistance reaches R_1 as $R_2 \to \infty$.

n Resistors in Parallel

Current flowing through n resistors $R_1, R_2, \ldots, R_n$ that are joined together in parallel is not unlike very large number of people wanting to tread across a mountain through a group of n tunnels that begin and end in very close proximity. And if the tunnels are of different widths, surely most people would choose to go through the tunnel that is the widest. And similar choices would be made for other tunnels. Same must also be the case for current flowing through n resistors in parallel so that most current would flow through that resistor that has the smallest resistance, and so on. Assuming the potential difference is V volts across the parallel assembly—which means the potential difference is V volts across each of the n resistors in the assembly—and currents $I_1, I_2, \ldots, I_n$ flow through the n resistors that are linked in parallel, then according to Ohm's law

$$V = I_1 R_1 = I_2 R_2 = \cdots = I_n R_n \ . \tag{9.12}$$

The above can be represented more compactly as

$$I_i = \frac{V}{R_i}, \ \ i = 1, 2, \ldots, n \ .$$

As a result the total current, $I_{\text{parallel}-\text{n}}(effective)$, the effective total resistance, $R_{\text{parallel}-\text{n}}(effective)$, and the potential V are related as follows:

$$\begin{aligned} I_1 + I_2 + \cdots + I_n &= \frac{V}{R_1} + \frac{V}{R_2} + \frac{V}{R_3} + \cdots + \frac{V}{R_n} \\ \equiv \ I_{\text{parallel}-\text{n}}(effective) &\equiv \frac{V}{R_{\text{parallel}-\text{n}}(effective)}. \end{aligned} \tag{9.13}$$

Thus

$$\frac{1}{R_{\text{parallel}-\text{n}}(effective)} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_n}. \tag{9.14}$$

This is an interesting result. To a layman, some of its consequences would be counter-intuitive. For instance, if all the n resistors were identical, each having resistance R, then according to (9.14)

$$\frac{1}{R_{\text{parallel}-\text{n}}(effective)} = \frac{n}{R},$$

or equivalently

$$R_{\text{parallel}-\text{n}}(effective) = \frac{R}{n}. \tag{9.15}$$

In other words, for very large number of resistors, all in parallel, the effective total resistance is very small.

A Few Resistors Together

Process (1) : A group of three resistors R_1, R_2, R_3 ohms—to be called the 'original trio'—is connected in series across an on–off switch and a battery operating at constant voltage V volts—see Fig. 9.3. Turning the switch on makes it a closed-loop circuit.

Process (2) : Three additional resistors of strength R_1 ohms each are connected in parallel to the resistor R_1 of the original trio : thus forming a sextet.

Process (3) : The sextet is enhanced further to form a nonet by connecting in parallel, to the resistor R_2 of the original trio, three new resistors of strength R_2 each.

Process (4) : In a fashion similar to the above, another three resistors of strength R_3 ohms each are added next. Each of these new resistors R_3 is connected in parallel to the resistor R_3 of the original trio. This process results in forming a duodectet to be called dd, composed of a total of twelve resistors, a switch, and a battery.

Problem (A)

Calculate the total effective resistance, $R_{\text{effect}-\text{dd}}$, of the duodectet dd demonstrated in Fig. 9.3.

9.1.4 Solution (A)

The four resistors R_1 in parallel can equivalently be replicated by a single resistor $R_{\text{effect}-\text{dd1}}$.

$$R_{\text{effect}-\text{dd1}} = \frac{R_1}{4} \ . \tag{9.16}$$

Similarly, the four resistors R_2 in parallel are replaced by a single resistor $R_{\text{effect}-\text{dd2}}$.

$$R_{\text{effect}-\text{dd2}} = \frac{R_2}{4} \ . \tag{9.17}$$

And finally, the four resistors R_3 in parallel can be replaced by a single resistor $R_{\text{effect}-\text{dd3}}$.

$$R_{\text{effect}-\text{dd3}} = \frac{R_3}{4} \ . \tag{9.18}$$

In the manner described above, the dd may be replaced by a new triplet. The effective resistance of such triplet would be $R_{\text{effect}-\text{dd}}$.

$$\begin{aligned} R_{\text{effect}-\text{dd}} &= R_{\text{effect}-\text{dd1}} + R_{\text{effect}-\text{dd2}} + R_{\text{effect}-\text{dd3}} \\ &= \frac{R_1 + R_2 + R_3}{4} \ . \end{aligned} \tag{9.19}$$

Figure 9.4 describes a circuit with an on–off switch, a battery operating at V volts, and four resistors: R_1, R_2, R_3 and R_4.

Left part of the circuit refers to points W, X, and Z. There is a single resistor of strength R_1 between points W and X. Connected in series to R_1 are two resistors of strength R_2 ohms. These two resistors are set in parallel between points X and Z. As a result, the left part of the circuit—that is, $W \rightarrow X \rightarrow Z$—can equivalently be expressed as having only two resistors, R_1 and $\frac{R_2}{2}$, connected in series.

The part of the circuit on the right—that is, between the points W, Y, and Z—has seven resistors. Four are of strength R_4 ohms each, and they are connected in parallel to each other. The other three are of strength R_3 ohms each, and they too are joined together in parallel to each other. Accordingly, this part of the circuit can also be expressed as having only two resistors that are connected in series. Their strengths are $\frac{R_4}{4}$, from $W \rightarrow Y$, and $\frac{R_3}{3}$, from $Y \rightarrow Z$.

In the manner described above, Fig. 9.5 effectively replicates Fig. 9.4. When the switch is on, the battery—operating at constant voltage V volts—drives current I_1 amps through resistors R_1 and $\frac{R_2}{2}$, and I_2 amps through resistors $\frac{R_4}{4}$ and $\frac{R_3}{3}$.

9.1.5 Problem (B)

In the circuit shown in Fig. 9.5, turn the switch on and work out currents I_1 and I_2, the potential difference $V_X - V_Y$, and the effective resistance, $R_{\text{effect}-\text{XY}}$ between points X and Y. The relevant data is as follows:

$$\begin{aligned} &V = 10 \text{ volts} ; \\ &R_1 = 15 \times 10^3 \text{ ohms} ; \quad R_2 = 70 \times 10^3 \text{ ohms} ; \\ &R_3 = 60 \times 10^3 \text{ ohms} ; \quad R_4 = 20 \times 10^3 \text{ ohms} . \end{aligned} \tag{9.20}$$

V is the battery voltage, and $R_1 \rightarrow R_4$ are the resistors described above. Their strength is as specified in (9.20).

9.1.6 Solution (B)

Shown in Fig. 9.5 is current flow, $= I \ amps$, from point W to point Z.

$$I = I_1 + I_2, \tag{9.21}$$

$$I_1 \;=\; \frac{V}{R_1 + \frac{R_2}{2}} \;, \quad I_2 \;=\; \frac{V}{\frac{R_3}{3} + \frac{R_4}{4}} \;. \tag{9.22}$$

The potentials at points X and Y are

$$\begin{aligned} V_X &= I_1 \left(\frac{R_2}{2}\right) \;, \\ V_Y &= I_2 \left(\frac{R_3}{3}\right) \;. \end{aligned} \tag{9.23}$$

[Note : Prove $V_X = V - R_1\, I_1$ and $V_Y = V - \left(\frac{R_4}{4}\right)\, I_2$ also lead to the same result as (9.24).] The effective potential difference, $V_{\text{effect}-\text{XY}}$, between X and Y is

$$V_{\text{effect}-\text{XY}} \;=\; V_X \;-\; V_Y \;=\; \frac{V\, R_2}{2\, R_1 + R_2} - \frac{V\, R_3}{R_3 + \frac{3\, R_4}{4}} \;. \tag{9.24}$$

Following (9.24) and (9.22), and using the numbers provided in (9.20), leads to

$$\begin{aligned} V_{\text{effect}-\text{XY}} &= (0.7 - 0.8)\ \text{volt} \;=\; -\,1\ \text{volt}\ ; \\ I_1 \;=\; 0.2 \times 10^{-3}\ &\text{amps};\ I_2 \;=\; 0.4 \times 10^{-3}\ \text{amps}. \end{aligned} \tag{9.25}$$

Knowing the effective potential difference, $V_{\text{effect}-\text{XY}}$, and the currents I_1 and I_2, the effective resistance, $R_{\text{effect}-\text{XY}}$, between points X and Y can now be calculated. The effective current flow from X to Y is $I_{\text{effect}-\text{XY}}$.

$$I_{\text{effect}-\text{XY}} \;=\; I_1 - I_2 \;=\; -\,(0.2) \times 10^{-3}\ \text{amps} \;. \tag{9.26}$$

Therefore

$$R_{\text{effect}-\text{XY}} \;=\; \frac{V_{\text{effect}-\text{XY}}}{I_{effect-XY}} \;=\; 5 \times 10^3\ \text{ohms} \;. \tag{9.27}$$

9.1.7 Problem (C)

Figure 9.6 shows a circuit that represents a slight change from the circuit in Fig. 9.5. Here the battery is missing, the on–off switch is absent, and the labels i_1 and i_2 have been removed. Thereby, in effect, connecting directly the points W and Z. [This effective double point is called the point WZ.] Calculate effective resistance between X and WZ and between WZ and Y.

9.1.8 Solution (C)

Detailed behavior of this 'slightly changed' circuit—shown in Fig. 9.6—is substantially different from that of its parent circuit 1e. Here the point W is in fact coincident with point Z, making the two a single point WZ.

Each of the couplets $(R_1 + \frac{R_2}{2})$ and $(\frac{R_3}{3} + \frac{R_4}{4})$ are now in parallel. Accordingly, the resistances between points X and WZ—that is $R_{totalX-WZ}$—and that between WZ and Y, are

$$R_{totalX-WZ} = R_1 \parallel (R_2/2) \quad , \tag{9.28}$$

and

$$R_{totalWZ-Y} = (R_4/4) \parallel (R_3/3) \quad . \tag{9.29}$$

The data provided in (9.20) yields

$$\begin{aligned} R_{totalX-WZ} &= \frac{21}{2} \times 10^3 \text{ ohms} \, . \\ R_{totalWZ-Y} &= 4 \times 10^3 \text{ ohms} \quad . \end{aligned} \tag{9.30}$$

9.2 Infinite Networks of Resistors

Resistance and Current

Work out the effective total resistance and currents in infinite assemblies of resistors described below. Consider first the circuit shown in Fig. 9.7.

Figure 9.7 is an infinite network of resistors, each of strength R ohms, connected together as follows.

Process (1) : Three resistors R, to be called the first-trio, are connected in series across a large battery operating at V volts and an on–off switch. Turning the switch on creates a closed-loop circuit.

Process (2) : The middle resistor R of the first-trio is connected in parallel to a similar second-trio.

Process (3) to Process (∞) : Process (2) is repeated ad infinitum.

When the switch is turned on, the battery notices only a single effective resistance $R_{\text{effective}}$ ohms. $R_{\text{effective}}$ is the sum of two resistors in series—each of strength R ohms—and two resistors, R and $R_{\text{effective}}$, in parallel. Thus

$$R_{\text{effective}} \; = \; 2R + R \parallel R_{\text{effective}} \, . \tag{9.31}$$

Equation (9.31) leads to a quadratic,

$$R^2_{\text{effective}} = 2R * R_{\text{effective}} + 2R^2, \tag{9.32}$$

which has one positive solution

$$R_{\text{effective}} = R\left(1 + \sqrt{3}\right) \ . \tag{9.33}$$

With the switch on, the battery operating provides current I which distributes itself into $I_1, I_2 \ldots I_\infty$ as it flows through an infinite array of resistors each of which is of strength R ohms.

Physical behavior of this circuit effectively replicates that of Fig. 9.7. The battery experiences an effective total resistance $R_{\text{effective}}$ and drives current I.

Ampere's law can now be used to determine the effective total current driven through the single effective resistance $R_{\text{effective}}$.

$$I_{\text{effective}} \ = \ \frac{V}{R_{\text{effective}}} \ . \tag{9.34}$$

An infinitely long repetitive array is not affected if a finite part is extracted. Therefore, the physics of the circuit in Fig. 9.7 is exactly replicated by that of circuit in Fig. 9.8. As shown, when current I reaches point A, it splits between two resistors R and $R_{\text{effective}}$. From point A to point B, current I_1 flows through resistor R, while the remainder, $\text{I} - I_1$, flows through the resistor $R_{\text{effective}}$. Both these currents are driven under the same potential difference. Therefore

$$I_1 \, R \ = \ [I - I_1] \, R_{\text{effective}} \ . \tag{9.35}$$

Combining this result with (9.33) and (9.34) leads to

$$I_1 \ = \ \left[\frac{R_{\text{effective}}}{R + R_{effective}}\right] I \ = \ \frac{V/R}{2 + \sqrt{3}} \ . \tag{9.36}$$

Next consider an infinite network of resistors shown in Fig. 9.9. The battery operates at constant voltage V across an on–off switch and a trio of resistors R, R_0, R. This is the first-trio of this network. The middle resistor of the first-trio, namely resistor R_0, is connected in parallel to an identical second-trio R, R_0, R, and the process is repeated ad infinitum. Current flow and the effective resistance of are treated next.

In the network shown in Fig. 9.10, current I amps distributes itself into $I_1, I_2 \ldots I_\infty$ as it flows through the infinite array of resistors. In order to work out the current I_1, the total effective current I_{effect0}, the total effective resistance R_{effect0}, one can follow the following procedure. Connect the midpoint resistor R_0 in parallel to a new resistor of strength R_{effect0}. And, as shown in Fig. 9.10, stop there. The upshot again is that the resistance of this assembly of four resistors should exactly be equal to the effective resistance, R_{effect0}, of the infinite network. That is,

$$R_{\text{effect0}} = R + R + R_0 \parallel R_{\text{effect0}} \ . \tag{9.37}$$

The above is a quadratic in R_{effect0} with two solutions: one positive and another un-physical solution that is negative. The physical solution is

$$R_{\text{effect0}} = R\left[1 + \sqrt{1 + 2\left(\frac{R_0}{R}\right)}\ \right] \ . \tag{9.38}$$

Some features of the effective resistance, R_{effect0}, are worth noting.

First: In the limit $R_0 \to R$, $R_{\text{effect0}} \to R[1 + \sqrt{3}]$. This—according to (9.33)—is as it should be.

Second: In the limit $R_0 \to \infty$, $R_{\text{effect0}} \to \infty$. Again this makes sense because now the total network consists of two infinite strings of resistors R that are unconnected—accept perhaps at ∞ !—and each of the strings has infinite resistance.

Third: The limit $R_0 \to 0$, where $R_{\text{effect0}} \to 2\,R$. To understand this result directly from Fig. 9.9 requires a little visual alacrity. In contrast, Fig. 9.10 is much clearer. When current I arrives at the point A, it sees two options for crossing over to point B. First, via zero resistance. Or second, through a finite resistance R_{effect0}. Quite sensibly, all of the current chooses to travel through zero rather than finite resistance. Hence, R_{effect0} is only $R + R$.

9.2.1 Current Flow

When the switch is on, current I flows out of the positive terminal of the battery and, after completing its journey, returns to the negative terminal. The total effective resistance of the circuit is R_{effect0} given in (9.38). Therefore, the total effective current is

$$I \ = \ \frac{V}{R_{\text{effect0}}} \ = \ \frac{V}{R\left[1 + \sqrt{1 + 2\left(\frac{R_0}{R}\right)}\ \right]} \ . \tag{9.39}$$

9.2.2 Current I_1

When current I reaches the point A in Fig. 9.10, it splits between two resistors R_0 and R_{effect0}. From point A to point B, current I_1 flows through resistor R_0, while the remainder, $I - I_1$, flows through resistor R_{effect0}. Both these currents flow under the same potential difference. Therefore

$$I_1\, R_0 \ = \ [I - I_1]\, R_{\text{effect0}} \tag{9.40}$$

leading to the result

$$I_1 \;=\; \left[\frac{R_{\text{effect0}}}{R_0 + R_{effect0}}\right] I \;=\; \frac{V}{R_0 + R\left[1 + \sqrt{1 + 2\left(\frac{R_0}{R}\right)}\right]} \,. \tag{9.41}$$

Next consider an infinite network of resistors shown in Fig. 9.11. The battery operates at constant voltage V volts across an on–off switch and a trio of resistors R_1, R_2, R_3. This is the first-trio. The middle resistor of the first-trio, namely resistor R_2, is connected in parallel to an identical second-trio R_1, R_2, R_3. And this process is repeated similarly ad infinitum. The effective resistance and the effective current flow are treated below.

The requirement that the effective resistance of the infinite circuit be the same as R_{NORC} leads to the equality

$$R_{\text{NORC}} \;=\; R_1 + R_3 + R_2 \parallel R_{\text{NORC}} \,. \tag{9.42}$$

As a quadratic it has two solutions: one of which is positive and the other is negative and therefore un-physical.

$$R_{\text{NORC}} \;=\; \frac{(R_1 + R_3)}{2} + \sqrt{\left[\left(\frac{R_1 + R_3}{2}\right)^2 + R_2\,(R_1 + R_3)\right]} \,. \tag{9.43}$$

Some notable features of R_{NORC} are its results in certain limits.

(1) $R_1 \to R_2 \to R_3 \equiv R$—meaning when all resistors are equal.

$$R_{\text{NORC}} \;\to\; R\left(1 + \sqrt{3}\right) \,. \tag{9.44}$$

This result, according to (9.33), is correct.

(1) : $R_3 \to 0$—meaning, when the lower line in the infinite circuit is a perfectly conducting wire—the result is

$$R_{\text{NORC}} \;=\; \frac{R_1}{2} + \sqrt{\left[\left(\frac{R_1}{2}\right)^2 + R_1\,R_2\right]} \,. \tag{9.45}$$

(2) : When $R_3 \to 0$ and $R_2 \to R_1$,

$$R_{\text{NORC}} \;=\; \frac{R_1}{2}\left[1 + \sqrt{5}\right] \,. \tag{9.46}$$

9.2.3 Current Flow

When the switch is on, current I amps flows out of the positive terminal of the battery and, after completing its journey, returns to the negative terminal. The total effective resistance of the circuit is R_{NORC}—see (9.43). Therefore, the total effective current is

$$\begin{aligned} I &= \frac{V}{R_{\text{NORC}}} \\ &= \frac{V}{\frac{(R_1+R_3)}{2} + \sqrt{\left[\left(\frac{R_1+R_3}{2}\right)^2 + R_2\,(R_1+R_3)\right]}}\,. \end{aligned} \tag{9.47}$$

[After leaving the positive terminal of the battery, the current I arrives at point A. There it splits into two parts. Current I_1 amps travels across the resistor R_2 to point B. The remainder, $(I - I_1)$ amps, travels through R_{NORC}. At point B, currents I_1 and $(I - I_1)$ join up so that their sum, namely the current I, returns to the battery. Because the currents flowing through the resistors, R_2 and R_{NORC}, are driven by the same voltage difference, the following relationship must obtain.

$$I_1\,R_2 \;=\; [I - I_1]\,R_{\text{NORC}} \tag{9.48}$$

With the help of (9.47), one can write (9.48) as

$$I_1 \;=\; \left[\frac{R_{\text{NORC}}}{R_2 + R_{\text{NORC}}}\right] I \;=\; \frac{V}{R_2 + R_{\text{NORC}}}\,. \tag{9.49}$$

Figure 9.13 portrays an infinite network which is a somewhat more involved network than those considered before. Shown on the left are three resistors in series, each of strength R_1 ohms. These three resistors are referred to as the first-triplet. The first-triplet and an on–off switch are connected across a large battery that operates at V volts. Turning the switch on makes it a closed-loop circuit.

The middle resistor R_1 of the first-triplet is connected to another three resistors each of which is of strength R_2 ohms. These six resistors make up the first-sextet. Following the first-sextet is a second-sextet that replicates the first-sextet. The second-sextet is followed by an identical third-sextet, and this process of adding sextet after sextet is repeated ad infinitum to form the infinite network. The effective resistance is—see Fig. 9.14—R_{eff}. The effective total current, I_{eff}, breaks up into infinite number of different parts which flow through the given infinite circuit. According to Ohm's law, $I_{\text{eff}} = \frac{V}{R_{\text{eff}}}$.

9.2.4 Effective Total Resistance

Calculate the effective total resistance of the infinite network drawn in Fig. 9.13. Here it is helpful to proceed in four successive stages.

To begin with notice the equivalence of Figs. 9.13 and 9.14.

First Stage: Treat the first-sextet as distinct from the rest. Focus on its second-triplet that is composed of three resistors of strength R_2 ohms each. Next, connect the middle resistor R_2 of this second-triplet in parallel to a new resistor of strength R_{eff}. This process has now formed a septet as shown in Fig. 9.14.

Second Stage: However, as shown in Fig. 9.15, this septet of resistors can again be reduced further to an equivalent-sextet by representing the last two resistors—that is, R_2 connected in parallel to R_{eff}—by a single resistor R_0. And because of the parallel nature of this connection, the relevant relationship is

$$R_0 \;=\; R_2 \parallel R_{\text{eff}} \quad . \tag{9.50}$$

Notice this relationship has two unknowns: R_0 and R_{eff}. To determine these two unknowns, one needs also a second relationship. For that purpose, proceed as follows.

Third Stage: The so-named equivalent-sextet achieved in the second stage described above can be reduced now to an equivalent-triplet. This is done by focussing on the resistor in the middle of the first-trio of resistors of strength R_1. Because this resistor is aligned in parallel to three other resistors, R_2, R_0, and R_2, all four can be replaced by a single equivalent resistance—see Fig. 9.16—R_q. The relevant relationship is

$$R_q \;=\; R_1 \parallel (R_0 + 2\,R_2) \quad . \tag{9.51}$$

As shown in Fig. 9.16, we are now left with a simple circuit: consisting of the power source and three resistors—namely, R_1, R_q, and R_1—that are connected in series.

Fourth Stage: Finally replace these three resistors by an equivalent-singlet: a single effective resistor, $R_{effect-Cc}$, that is connected to the power source—see Fig. 9.17. This means

$$R_{effect-Cc} \;=\; R_1 + R_q + R_1. \tag{9.52}$$

9.2.5 The Result for $R_{effect-Cc}$

There are a total of three equations—namely, (9.50), (9.51), and (9.52)—and three unknowns: namely, $R_{effect-Cc}$, R_0, and R_q. After combining (9.50), (9.51), and (9.52), straightforward algebra is used to calculate $R_{effect-Cc}$ as a function of R_1 and R_2. However, even without examining the result in detail, one can predict its outcome in three particular limits: (1), (2), (3).

(1) Consider the resistance notated R_q. In the limit $R_2 -> \infty$, (9.51) gives $R_q -> R_1$. As such (9.52) would lead to

$$R_{effect-Cc} -> 3\,R_1 \quad . \tag{9.53}$$

(2) Next consider R_0. In the opposite limit, namely $R_2 -> 0$, (9.50) leads to

$$R_0 -> 0 \quad . \tag{9.54}$$

(3) And with R_0 and R_2 both tending to zero, (9.51) leads to $R_q -> 0$. As a result (9.52) yields

$$R_{effect-Cc} -> 2\,R_1 \quad . \tag{9.55}$$

9.2.6 $R_{effect-Cc}$ as a Function of R_1 and R_2

For convenience, introduce the notation

$$x \;=\; \left(\frac{R_2}{R_1}\right) \quad . \tag{9.56}$$

After some straightforward but time-consuming algebra, $R_{effect-Cc}$ can be calculated by combining (9.50), (9.51), and (9.52). The result is

$$R_{effect-Cc} \;=\; R_1 \frac{\left[1 + 4x - x^2 + \sqrt{1 + 10x + 26x^2 + 10x^3 + x^4}\right]}{(1 + 3x)} \quad . \tag{9.57}$$

It is helpful to have a few simple tests that check the accuracy of (9.57).

(a) : To that end, first try $R_2 -> \infty$. In that limit, according to (9.56), $x -> \infty$. Therefore (9.57) transforms as follows.

$$\begin{aligned}\left[R_{effect-Cc}\right]_{x\,->\infty} &= R_1 \frac{\left[4x - x^2 + x^2\,(1 + 5/x)\right]}{(3x)} \\ &= R_1 \frac{\left[4x - x^2 + x^2 + 5x\right]}{(3x)} \;=\; 3R_1 \quad .\end{aligned} \tag{9.58}$$

This is the result in (1) above.

(b) : Next, look at the case $R_2 -> 0$. That implies $x -> 0$. In this limit (9.57) leads to

$$\left[R_{effect-Cc}\right]_{x\ ->0} = R_1 \frac{1+\sqrt{1}}{1} \ = \ 2\,R_1 \quad . \tag{9.59}$$

(c) : Finally, consider $R_2 \ = \ R_1$: that is, $x = 1$. Setting $x = 1$ in (9.57) correctly leads to the result obtained earlier in (9.33).

$$\left[R_{effect-Cc}\right]_{x\ ->1} = R_1 \frac{\left[4+\sqrt{48}\right]}{4} \ = \ R_1 \left[1+\sqrt{3}\right] \quad . \tag{9.60}$$

9.2.7 Table 1 : Electric Circuit Elements

Whenever convenient, some of the following notation may be used. Also, to indicate time dependence, symbols may be in lower case.

$$\left|\begin{array}{c} Notation\ Used \\ \\ Capacitance,\ C \quad Measured\ in\ farads \quad : F \\ Charge,\ Q \quad Measured\ in\ coulombs \quad : C \\ Current,\ I \quad Measured\ in\ amperes \quad : A \\ Inductance,\ L \quad Measured\ in\ henries \quad : H \\ Resistance,\ R \quad Measured\ in\ ohms \quad : \Omega \\ Voltage,\ V \quad Measured\ in\ volts \quad : V \end{array}\right| \tag{9.61}$$

9.3 Inductors

A coil made of several turns of a very good conducting wire is the physical equivalent of an inductor (L). If constant current $i(0)$ flows through (L), the coil acts merely as a very week resistor. According to Ampere's law, constant current produces a time-independent magnetic field whose magnitude is proportional to the current. On the other hand, if the current $i(t)$ is changing in time, it produces a changing magnetic field. And the resulting change in the magnetic flux induces an electro-motive force (emf). The (emf) produced is proportional to the rate of change of the current.

Because of the weakness of the internal resistance, current flows that are constant in time mostly produce un-substantial changes in voltage across an inductor. On the other hand, when current flow is changing in strength, say at rate $\left(\frac{\mathrm{d}\,i(t)}{\mathrm{d}t}\right)$ amperes per second, an inductor of strength L henrys affects nonzero potential drop $v(t)$ across it.

$$v(t) \ = \ L \frac{\mathrm{d}\,i(t)}{\mathrm{d}t} \quad . \tag{9.62}$$

9.3.1 Two Inductors in Series

Imagine time-dependent current $i(t)$ amps flowing through two inductors of strength L_1 and L_2 henries connected together in series. The drop in potential across the first inductor is $L_1 \frac{\mathrm{d}i(t)}{\mathrm{d}t}$ and that across the second is $L_2 \frac{\mathrm{d}i(t)}{\mathrm{d}t}$. And because these drops in potential occur in series, the total drop in potential is their sum. The same result would obtain if the given current passed through a single inductor of strength $(L_1 + L_2)$ henries.

$$L_1 \frac{\mathrm{d}\, i(t)}{\mathrm{d}t} + L_2 \frac{\mathrm{d}\, i(t)}{\mathrm{d}t} = (L_1 + L_2) \frac{\mathrm{d}\, i(t)}{\mathrm{d}t} \quad . \tag{9.63}$$

In other words, when placed in series the inductances add much the same way as do resistances.

9.3.2 Two Inductors in Parallel

Imagine time-dependent current $i(t)$ amps flowing through two inductors of strength L_1 and L_2 henries connected together in parallel. The drop in potential across the first inductor is $L_1 \frac{\mathrm{d}i(t)}{\mathrm{d}t}$ and that across the second is $L_2 \frac{\mathrm{d}i(t)}{\mathrm{d}t}$. And because these drops in potential occur in parallel, they are the same.

$$L_1 \frac{\mathrm{d}\, i_1(t)}{\mathrm{d}t} = L_2 \frac{\mathrm{d}\, i_2(t)}{\mathrm{d}t} \equiv \text{Const} \quad . \tag{9.64}$$

With slight manipulation of (9.64), one can write

$$\begin{aligned} \frac{\mathrm{d}\, i_1(t)}{\mathrm{d}t} &= \frac{\text{Const}}{L_1} \quad ; \\ \frac{\mathrm{d}\, i_2(t)}{\mathrm{d}t} &= \frac{\text{Const}}{L_2} \quad . \end{aligned} \tag{9.65}$$

Denote the total current i_{parll} and total inductance L_{parll} and use (9.65) to write

$$\frac{\mathrm{d}\, i_{\text{parll}}}{\mathrm{d}t} \equiv \frac{\mathrm{d}\, i_1(t)}{\mathrm{d}t} + \frac{\mathrm{d}\, i_2(t)}{\mathrm{d}t} = \frac{\text{Const}}{L_1} + \frac{\text{Const}}{L_2} \equiv \frac{\text{Const}}{L_{\text{parll}}} \quad . \tag{9.66}$$

Equation (9.66) leads to the following relationship obeyed by the total inductance, L_{parll}, of two inductors, L_1 and L_2, added together in parallel.

$$\frac{1}{L_{\text{parll}}} = \frac{1}{L_1} + \frac{1}{L_2} \quad . \tag{9.67}$$

Once again, the above is similar to the behavior of resistors.

9.3.3 Infinite Network of Inductors

Consider the infinite network of inductors $L_1,\ L_2,\ L_3$ shown in Fig. 9.18. Its effective inductance may equivalently be represented by the inductance of a single inductor of strength L_{eff}. As a result, inductance L_{eff} may be equated to the inductance of the four inductors shown in Fig. 9.19. That is,

$$L_{\text{eff}} \ = \ L_1 + L_2 + L_3 \parallel L_{\text{eff}} \ . \tag{9.68}$$

Equivalently

$$L_{\text{eff}}^2 \ = \ L_{\text{eff}}\,(L_1 + L_2) + L_3\,(L_1 + L_2) \ . \tag{9.69}$$

This is a quadratic with a positive and a negative solution. The physically acceptable one is the positive solution

$$L_{\text{eff}} \ = \ \frac{(L_1 + L_2)}{2}\left[1 + \sqrt{1 + \frac{4\,L_3}{(L_1 + L_2)}}\right] \tag{9.70}$$

Because in the limit $L_3 \to 0$ points A and B get connected, L_{eff} must $\to L_1 + L_2$— exactly as suggested by (9.70).

9.3.4 Capacitors

Two conducting plates, separated by dielectric constitute a parallel plate capacitor (C). [Note, a good dielectric conducts electricity very poorly but supports electric fields well.] Equal amounts of opposite charge may be stored on opposing plates of (C). If these charges are $+Q$ and $-Q$, and the potential difference between the opposing plates is V, the capacitance is C.

$$C \ \text{farads} \ = \ \frac{Q \ \text{coulombs}}{V \ \text{volts}} \ . \tag{9.71}$$

As such, a farad is equal to coulombs per volt: or, equivalently, (amperes. second) per volt. Also, a coulomb is equivalent to farads. volt.

9.3.5 Charging a Capacitor

Assume a large battery that can supply current at fixed voltage V is connected to a capacitor (C), a perfect resistor (R), and a switch. [Note : A perfect resistor obeys

Ohm's law exactly and contains no stray inductance or capacitance.] These items are placed in series forming a closed-loop circuit. The charging of (C) requires an inflow of current into, say, the left plate and an equal outflow from the right plate. Generally this process is not instantaneous. Rather, the charging occurs exponentially with a relaxation time that is representative of the nature of the capacitor as well as details of the resistance in the circuitry. And both the current i as well as the charge q are functions of time. Note, current and charge are related because one represents the rate of flow of the other. That is,

$$i(t) \;=\; \frac{\mathrm{d}q(t)}{\mathrm{d}t}\;. \tag{9.72}$$

Therefore at some specified time t, the charge $q(t)$ that has accumulated on the left plate is

$$q(t) \;=\; \int_{-\infty}^{t} \left(\frac{\mathrm{d}q(t)}{\mathrm{d}t}\right) \mathrm{d}t \;=\; \int_{-\infty}^{t} i(t)\,\mathrm{d}t \quad . \tag{9.73}$$

Of course, at that time t, the charge on the right plate is $-q(t)$.

9.3.6 *Two Capacitors in Parallel*

Imagine two capacitors, C_1 and C_2, connected in parallel. This implies the left plates of the two capacitors are connected together as are the right plates. As a result, both capacitors experience the same voltage difference, say V volts, leading to the relationships

$$Q_1 \;= C_1 \cdot V \;\;;\;\; Q_2 \;= C_2 \cdot V \;\;;\;\; Q_1 + Q_2 \;=\; (C_1 + C_2) \cdot V \quad , \tag{9.74}$$

where Q_1 and Q_2 are the charges held on the positive plates of capacitors C_1 and C_2, respectively. The total charge held is $Q_{\text{total}-\text{parallel}}$ and the total capacitance is $C_{\text{total}-\text{parallel}}$.

$$Q_{\text{total}-\text{parallel}} \;=\; Q_1 + Q_2 = (C_1 + C_2) \cdot V \;\equiv\; C_{\text{total}-\text{parallel}} \cdot V \quad . \tag{9.75}$$

Equation (9.75) leads to the result

$$C_{\text{total}-\text{parallel}} \;=\; (C_1 + C_2) \quad . \tag{9.76}$$

This is an interesting result in that it is similar to that of resistors being combined in series, rather than in parallel.

9.3.7 Two Capacitors in Series

Imagine two capacitors, C_1 and C_2, joined together in series. This implies the negative plate of capacitor C_1 is connected to the positive plate of capacitor C_2. As a result, the charges will be distributed as follows. If the left plate—meaning the positive plate—of capacitor C_1 has charge $+Q$, its right plate will have charge $-Q$. This will result in the left plate of capacitor C_2 with charge $+Q$ and its right plate with charge $-Q$. And if the potential drop across capacitor C_1 is V_1, and that across C_2 is V_2, the total potential drop across the two capacitors in series will be $V_{\text{total}-\text{series}-2} = V_1 + V_2$ and the following relationships will hold.

$$V_{\text{total}-\text{series}-2} \;=\; V_1 + V_2 \;=\; \frac{Q}{C_1} + \frac{Q}{C_2} \equiv \frac{Q}{C_{\text{total}-\text{series}-2}} \tag{9.77}$$

leading to the result

$$\frac{1}{C_{\text{total}-\text{series}-2}} \;=\; \frac{1}{C_1} + \frac{1}{C_2} \quad . \tag{9.78}$$

If the two capacitors are equal—that is, $C_1 = C_2 \equiv C$—their sum in series is equal to a half of each: meaning $C_{\text{total}-\text{series}-2} = \frac{C}{2}$.

The above results are similar to those of resistors being added in parallel. An interesting consequence of this is that when very large number of capacitors are added together in series, their sum is very small. For instance, the result of adding n capacitors of strength C in series is $\frac{C}{n}$, and if $n \to \infty$ the sum is zero.

9.3.8 Infinite Network of Capacitors

Consider the infinite network of capacitors $C_1,\ C_2,\ C_3$ shown in Fig. 9.20. Its effective capacitance may equivalently be represented by the capacitance of a single capacitor of strength C_{eff}. As a result, capacitance C_{eff} may be equated to the capacitance of the four capacitors shown in Fig. 9.21.

That is,

$$\frac{1}{C_{\text{eff}}} \;=\; \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3 + C_{\text{eff}}} \quad . \tag{9.79}$$

Equivalently

$$C_{\text{eff}}^2\,(C_1 + C_2) \;=\; -\,C_{\text{eff}}\,C_3\,(C_1 + C_2) + C_1 C_2 C_3 \quad . \tag{9.80}$$

This is a quadratic with a positive and a negative solution. The physically acceptable solution is the positive one, namely

$$C_{\text{eff}} \;=\; -\,\frac{C_3}{2} + \sqrt{\left(\frac{C_3}{2}\right)^2 + \frac{C_1 C_2 C_3}{C_1 + C_2}}\;. \tag{9.81}$$

9.3.9 Comment

Some of the implications of (9.81) are the following.

(a) In the limit $C_3 \to 0$, $C_{\text{eff}} \to 0$.

This result makes eminent sense because now the network is composed only of infinite numbers of capacitors, C_1 and C_2, connected in series. Equivalently, infinite numbers of resistors connected in parallel would have vanishing resistance.

(b) Essentially similar argument applies to the case when capacitors $C_1 \to 0$ because now the network resembles an infinite array of capacitors $C_0 \equiv \left(\frac{C_2.C_3}{C_2+C_3}\right)$ that are connected in series. Thus, one expects that in the limit $C_1 \to 0$, C_{eff} would also $\to 0$.

(c) Clearly, the same is also true if capacitors $C_2 \to 0$ because here the network resembles an infinite array of capacitors $C_{0'} \equiv \left(\frac{C_1.C_3}{C_1+C_3}\right)$ that are connected in series.

(d) Finally, if both capacitors C_1 and $C_2 \to 0$, the network would resemble an infinite array of C_3 capacitors that are connected in series. As a result $C_{\text{eff}} \to 0$.

(e) When all capacitors are equal, meaning $C_1 = C_2 = C_3 \equiv C$, according to (9.81) the effective capacitance of the corresponding infinite network in Fig. 9.20: Circuit ($CAPACIT$) would be C_{eff}.

$$C_{\text{eff}} \;=\; \frac{C}{2}\left[\sqrt{3} - 1\right]\;. \tag{9.82}$$

9.4 R-C Series-Circuit

Figure 9.22 displays a series circuit comprised of an on–off switch, a large battery, a resistor R, and a capacitor C. At time t the switch is on, the battery supplies time-dependent current $i(t)$ at constant voltage V, and the charges on the left and the right plates of the capacitor are $q(t)$ and $-q(t)$. The following differential equation describes this closed-loop circuit.

$$R\,i(t) + \frac{q(t)}{C} \;=\; V \;\equiv\; R\,\frac{\mathrm{d}q(t)}{\mathrm{d}t} + \frac{q(t)}{C}\;. \tag{9.83}$$

On the right-hand side of (9.83), $i(t)$ has been replaced by $\frac{\mathrm{d}q(t)}{\mathrm{d}t}$. Initially, meaning at $t \leq 0$, the capacitor does not have any charge on either of its plates and both the plates are at zero potential. However, instantly after the switch is turned on, current starts flowing, and the process of charging the capacitor gets initiated. As the charge begins accumulating on the plates—left side positive and the right side negative—potential difference between the plates begins increasing from its initial value of zero. As a result, the effective voltage across the resistor begins decreasing. This engenders time-dependent decrease of the current. The resultant charging of the plates is an exponential process, and eventually, the current stops flowing, the capacitor gets fully charged, and the potential difference across the plates of the capacitor equals the battery voltage V. It is convenient to divide both sides of (9.83) by R and re-write it as

$$\frac{\mathrm{d}q(t)}{\mathrm{d}t} + \frac{q(t)}{\tau_{R-C}} = \frac{V}{R} \tag{9.84}$$

where

$$\tau_{R-C} \equiv R\,C \tag{9.85}$$

is the time constant of the $R - C$ series circuit. When R is measured in Ω—ohms—and C in F—farads—τ_{R-C} is measured in seconds.

Recall that inhomogeneous ordinary differential equations with constant coefficients and their solutions were discussed in detail in (3.74)–(3.80). And (9.84) is a simple, one-dimensional such differential equation. Its solution is

$$q(t) = \sigma_0 \exp\left[-\left(\frac{t}{\tau_{R-C}}\right)\right] + C\,V \ . \tag{9.86}$$

The unknown constant σ_0 can be determined from one boundary condition.

9.4.1 Examples Group I

Problem

Assume the capacitor in the system described above and shown in Fig. 9.22 is in its uncharged state. Turn the switch on at time $t = t_0$ and work out the time dependence of the charge and the current.

Solution

Charge on the capacitor at time t_0 is zero. Equation (9.86) at $t = t_0$,

$$q(t_0) \;=\; 0 \;=\; \sigma_0 \exp\left[-\left(\frac{t_0}{\tau_{R-C}}\right)\right] \;+\; C\,V \;, \tag{9.87}$$

gives

$$\sigma_0 \;=\; -\,C\,V \exp\left[\left(\frac{t_0}{\tau_{R-C}}\right)\right] \;. \tag{9.88}$$

Accordingly, we have

$$q(t) \;=\; C\,V \left\{1 - \exp\left[-\left(\frac{t-t_0}{\tau_{R-C}}\right)\right]\right\} \tag{9.89}$$

and

$$i(t) \;=\; \frac{\mathrm{d}q(t)}{\mathrm{d}t} \;=\; \frac{V}{R} \exp\left[-\left(\frac{t-t_0}{\tau_{R-C}}\right)\right] \;. \tag{9.90}$$

Equation (9.90) predicts that within a time interval equal to τ_{R-C} the current drops to about one-third—actually, 36.7879441%— of its original value $\left(\frac{V}{R}\right)$, while, according to (9.89), the magnitude of the charge on either plate rises to about two-thirds of its maximum value, which is $C\,V$.

9.4.2 R-L Series Circuit

At time t, the switch is on and the battery supplies time-dependent current $i(t)$ at constant voltage V to a series connection constituted of a resistor R and an inductor L. The following differential equation describes this closed-loop circuit.

$$R\,i(t) + L\,\frac{\mathrm{d}i(t)}{\mathrm{d}t} \;=\; V. \tag{9.91}$$

Divide both sides by L and solve the above. The result is

$$i(t) \;=\; \sigma_0 \exp\left(-\frac{R}{L}\,t\right) + \frac{V}{R}. \tag{9.92}$$

9.4.3 Examples Group II

Consider a closed-loop circuit containing a large battery that can supply current at constant voltage V. Other items in the circuit are a resistor R, an inductor L, and a

switch. All these items are connected in series. Turn the switch on at time $t = t_0$ and work out the time dependence of the charge and the current.

9.4.4 Solution

At $t = t_0$, the current is vanishing. As such the solution (9.92) gives

$$i(t_0) \;=\; 0 \;=\; \sigma_0 \exp\left(-\frac{R}{L}\,t_0\right) + \frac{V}{R} \quad . \tag{9.93}$$

Or equivalently

$$\sigma_0 \;= -\left(\frac{V}{R}\right) \exp\left(\frac{R}{L}\,t_0\right) \quad . \tag{9.94}$$

Inserting the above value of σ_0 into (9.92) yields the result

$$i(t) \;=\; \left(\frac{V}{R}\right)\left[1 - \exp\left\{-\frac{(t-t_0)}{\tau_{R-L}}\right\}\right] \tag{9.95}$$

where τ_{R-L} is the $R-L$ system time constant.

$$\tau_{R-L} \;=\; \frac{L}{R}. \tag{9.96}$$

When R is measured in Ω—ohms—and L in H—henrys—τ_{R-L} is measured in seconds.

Note that within a time interval equal to τ_{R-L} the current drops to 63.212056% of its maximum value $\left(\frac{V}{R}\right)$.

9.4.5 L-C Series-Circuit

The following is a differential equation that describes a closed-loop circuit formed of a large battery that at time t supplies current $i(t)$ at constant voltage V to a series connection constituted of an inductor (L), a capacitor (C), and a switch that is on—see Fig. 9.23.

$$\frac{q(t)}{C} + L\,\frac{\mathrm{d}i(t)}{\mathrm{d}t} \;=\; V \quad . \tag{9.97}$$

Replacing $i(t)$ by $\frac{dq(t)}{dt}$ and dividing both sides by L lead to a second-order inhomogeneous linear ordinary differential equation for the charge $q(t)$ on the left plate of the capacitor. [The charge on the right plate is $-q(t)$.]

$$\frac{d^2q(t)}{dt^2} + \frac{q(t)}{(\tau_{L-C})^2} = \frac{V}{L} \ . \tag{9.98}$$

In (9.98)

$$(\tau_{L-C}) \equiv \sqrt{L\,C} \tag{9.99}$$

is the time constant of the L-C series circuit. When L is measured in H and C in F, τ_{L-C} is measured in seconds. [Note : $\sqrt{onehenry.onefarad}$= 1 second]

The solution to (9.98)—which is an inhomogeneous linear ordinary differential equation with constant coefficients—is found in the usual manner [Compare (3.55) to (3.80)].

$$\begin{aligned} q(t) &= \sigma_1 \sin(\omega t) + \sigma_2 \cos(\omega t) + C\,V \ ; \\ i(t) &= \frac{dq(t)}{dt} = \omega\,[\sigma_1 \cos(\omega t) - \sigma_2 \sin(\omega t)] \ . \end{aligned} \tag{9.100}$$

The charge $q(t)$ and the current $i(t)$ are periodic in time with angular frequency ω and frequency ν.

$$\omega = \frac{1}{\tau_{L-C}} = 2\pi\nu \ . \tag{9.101}$$

The unknown constants σ_1 and σ_2, in (9.100), can be determined by two boundary conditions.

9.4.6 Examples Group III

Problem

Imagine Fig. 9.23 without the battery. And consider an L-C series circuit that despite the absence of the battery has an on/off switch. The switch is kept off as long as the time t is $< t_0$. The charge $\pm Q_0$ on the plates stays put as is while the switch is off. When at $t = t_0$ the switch is turned on, it allows the charge to start flowing. Describe the resultant discharge of the capacitor.

Battery Absent

Solution

In the absence of the battery, the voltage V can be excluded from the $L - C$ series-circuit differential equation (9.98). The two unknown constants, σ_1 and σ_2, in the

result of the remaining equation are determined from the known boundary condition: $q(t_0) = Q_0$ and $i(t_0) = 0$.

$$\begin{aligned} q(t_0) &= Q_0 = \sigma_1 \sin(\omega t_0) + \sigma_2 \cos(\omega t_0) \ ; \\ i(t_0) &= 0 = \omega \left[\sigma_1 \cos(\omega t_0) - \sigma_2 \sin(\omega t_0)\right] \ . \end{aligned} \tag{9.102}$$

Thus

$$\begin{aligned} \sigma_1 &= Q_0 \sin(\omega t_0) \ ; \\ \sigma_2 &= Q_0 \cos(\omega t_0) \, . \end{aligned} \tag{9.103}$$

Inserting the results given in (9.103) into (9.100) yields

$$\begin{aligned} q &= Q_0 \cos\left[\omega (t - t_0)\right] \ ; \\ i &= -\omega \, Q_0 \sin\left[\omega (t - t_0)\right] \ . \end{aligned} \tag{9.104}$$

Accordingly, at time t, the voltage drop across the capacitor is

$$\frac{q}{C} = \left(\frac{Q_0}{C}\right) \cos\left[\omega (t - t_0)\right] \ . \tag{9.105}$$

And that across the inductor L is

$$L \frac{\mathrm{d}\, i}{\mathrm{d}t} = -\left(\omega^2 L \ Q_0\right) \cos\left[\omega (t - t_0)\right] \ = \ -\left(\frac{Q_0}{C}\right) \cos\left[\omega (t - t_0)\right] \ . \tag{9.106}$$

Because there is no impressed voltage, the sum of these two voltage drops is zero.

9.5 L-R-C Series Circuit

Constant Impressed Voltage

9.5.1 Examples Group IV

Problem

Consider an L-R-C circuit with a battery that operates at constant potential V. The switch is kept off as long as the time t is < 0. While the switch is off, the capacitor plates have no charge. When at $t = +0$ the switch is turned on, it allows charge to start flowing. Describe the charge $q(t)$ and its rate of accumulation.

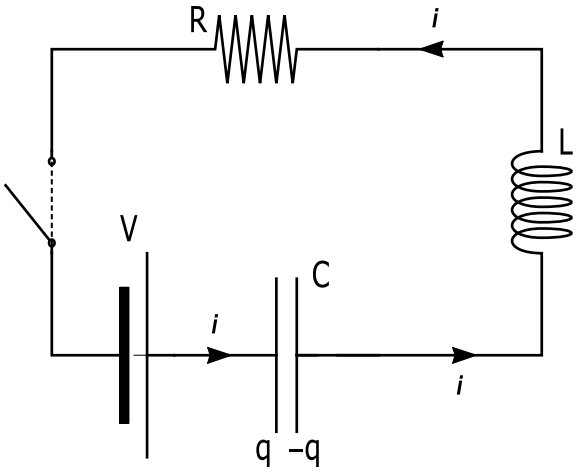


Fig. 9.24 R-L-C series-circuit, with a battery and on–off switch. At time t, charges on the capacitor plates are $\pm q(t)$ and current $i(t)$ flows through the circuit

9.5.2 Solution

The L-R-C series-circuit shown in Fig. 9.24 contains resistance R ohms, inductance L henries, and capacitance C farads. Shown also is the battery that provides constant voltage V volts. Also there is an on–off switch. Before the switch is turned on at t=0, the capacitor is completely uncharged. At $t = +0$, charge $+\mathrm{d}q$ streams out from the positive terminal of the battery and starts getting deposited on the left-hand plate of the capacitor. As a result, equal amount of negative charge—that is, $-\mathrm{d}q$—is induced on the right-hand plate of the capacitor. Because of the physical requirement that charge neither be created nor destroyed, this process results in equal amount of positive charge—that is $+\mathrm{d}q$—to move away from the right-hand plate of the capacitor, continue its travel around the circuit, and in time $\mathrm{d}t$ return to the negative terminal of the battery. This whole process is equivalent to current, $\frac{\mathrm{d}q}{\mathrm{d}t}$, flowing across the entire circuit, from the positive terminal of the battery back through to the negative terminal.

Streaming of the charge and changes in potential across the circuit are interconnected. According to Kirchhoff's second law, over a complete cycle, the total change in potential is zero. Assuming the battery provides current at V volts, a flowchart of the voltage changes across the whole circuit, at a given time $t \geq 0$, would be as follows. Voltage change across capacitor C + voltage change across inductor L + voltage change across resistor R + voltage change across the battery = 0. That is,

$$-\frac{q(t)}{C} - L\frac{\mathrm{d}i(t)}{\mathrm{d}t} - R\,i(t) + V \;=\; 0\,. \tag{9.107}$$

Divide (9.107) by L, replace $i(t)$ by $\frac{\mathrm{d}q(t)}{\mathrm{d}t}$, and rearrange slightly.

$$\frac{\mathrm{d}^2 q(t)}{\mathrm{d}t^2} + \left(\frac{R}{L}\right)\frac{\mathrm{d}q(t)}{\mathrm{d}t} + \left(\frac{1}{L\,C}\right) q(t) \;=\; \frac{V}{L} \quad . \tag{9.108}$$

The above is an inhomogeneous linear ordinary differential equation with constant coefficients C. Its solution, according to (3.59), is

$$q(t) \;=\; S_{\mathrm{comp}}(t) + I_{pi}(t) \tag{9.109}$$

where $S_{\mathrm{comp}}(t)$ is the complementary solution and $I_{pi}(t)$ is the particular integral that are defined as follows.

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \left(\frac{R}{L}\right)\frac{\mathrm{d}}{\mathrm{d}t} + \left(\frac{1}{L\,C}\right)\right] S_{\mathrm{comp}}(t) \;=\; 0 \quad . \tag{9.110}$$

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \left(\frac{R}{L}\right)\frac{\mathrm{d}}{\mathrm{d}t} + \left(\frac{1}{L\,C}\right)\right] I_{pi}(t) \;=\; \frac{V}{L} \quad . \tag{9.111}$$

According to the description given in detail in Chap. 3—see (3.55) and (3.56)—$S_{\mathrm{comp}}(t)$ is readily calculated. Similarly, $I_{pi}(t)$ may be found by using (3.61) and (3.67). We get

$$\begin{aligned} S_{\mathrm{comp}}(t) \;=\; & \sigma_1 \exp\left(-\frac{R\,t}{2\,L}\right) \exp\left\{t\sqrt{\left(\frac{R}{2\,L}\right)^2 - \left(\frac{1}{L\,C}\right)}\right\} \\ & + \sigma_2 \exp\left(-\frac{R\,t}{2\,L}\right) \exp\left[-\left\{t\sqrt{\left(\frac{R}{2\,L}\right)^2 - \left(\frac{1}{L\,C}\right)}\right\}\right] \quad . \end{aligned} \tag{9.112}$$

$$I_{pi}(t) \;=\; \left(\frac{V}{L}\right)\frac{1}{\left(\frac{1}{L\,C}\right)} \;=\; C\,V \quad . \tag{9.113}$$

It is interesting to note that in (9.110) and (9.111) , the expression $\left(\frac{1}{L\,C}\right)$ plays the role of angular velocity squared, while $\left(\frac{R}{L}\right)$ acts as a friction coefficient. [Note : In (9.112) the constants σ_1 and σ_2 are arbitrary and, as usual, can be determined by two boundary conditions.] For convenience, introduce the notation

$$\alpha \;=\; \sqrt{\left(\frac{R}{2\,L}\right)^2 - \left(\frac{1}{L\,C}\right)} \,. \tag{9.114}$$

Using the above notation, re-write (9.112) as

$$S_{\mathrm{comp}}(t) \;=\; \sigma_1 \,\exp\left(\varpi_1 t\right) + \sigma_2 \,\exp\left(\varpi_2 t\right). \tag{9.115}$$

where

$$\varpi_1 = \left(\frac{-R}{2L}\right) + \alpha\,,$$
$$\varpi_2 = \left(\frac{-R}{2L}\right) - \alpha\ . \tag{9.116}$$

No Impressed Voltage

When the battery is absent, the impressed voltage, V, is zero. As a result, $I_{pi}(t)$ is vanishing and according to (9.109) and (9.115)

$$q(t) \;=\; S_{\text{comp}}(t) \;=\; \sigma_1\,\exp(\varpi_1 t) + \sigma_2\,\exp(\varpi_2 t). \tag{9.117}$$

9.5.3 Over-Damped Series Circuit

Capacitor Charge $q(t)$

The circuit is over-damped when the resistance R is overpowering and, as a result, α—see (9.114)—is real rather than imaginary. This results in the discriminant being greater than zero, that is,

$$\alpha^2 \;=\; \left(\frac{R}{2L}\right)^2 - \left(\frac{1}{LC}\right) \;>\; 0\,, \tag{9.118}$$

which leads to another inequality

$$\left(\frac{R}{2L}\right)^2 \;>\; \alpha^2\,. \tag{9.119}$$

Because L, R, and C are all positive, (9.119) implies

$$\left(\frac{R}{2L}\right) \;>\; \alpha \tag{9.120}$$

and (9.114) indicates that α is positive. Then, according to (9.114), (9.116), and (9.120), we have the inequalities

$$\begin{aligned} \varpi_1 &< 0\ ; \\ \varpi_2 &< 0\ ; \\ \varpi_1 - \varpi_2 &> 0\ . \end{aligned} \tag{9.121}$$

In view of these inequalities, (9.117) predicts that irrespective of whether σ_1 and σ_2 are positive or negative, as time passes the charge $q(t)$ must exponentially decrease, heading to zero as $t- > \infty$.

9.5.4 Current Flow Across Capacitor

Shown in Fig. 9.25 is a closed-loop series circuit that contains a resistor, an inductor, a capacitor, and an on–off switch. The switch is off while $t \leq 0$. At $t = 0$, the charge on the left plate of the capacitor is $+q(0)$ and the current $i(0) = 0$. However, immediately as $t > 0$ the switch is turned on and the capacitor begins discharging: thereby resulting in current $i(t)$ flowing from the positively charged left plate all the way around the circuit to the negatively charged right plate of the capacitor.

Problem

Work out the time dependence of the capacitor charge q and the electric current i in the circuit. For numerical calculation use:
$Inductance\ L\ =\ 1\ H\ ;\ Resistance\ R\ =\ 10^3\ \Omega\ ;$
$Capacitance\ C\ =\ \left(9 \cdot 10^4\right)^{-1}\ F\ ;\ Capacitor\ Charge\ q(0)\ =\ 10^{-3}\ C$

Solution

Clearly $I_{pi}(t) = 0$ for $t \geq 0$ because there is no impressed voltage.

Regarding the charge $q(t)$ on the capacitor, and the current $i(t)$ flowing in the circuit, at $t = 0$ we are told $i(0) = 0$ and according to (9.115)

$$S_{\text{comp}}(t = 0)\ =\ q(0)\ =\ \sigma_1 + \sigma_2\ =\ 10^{-3}\ C\ . \tag{9.122}$$

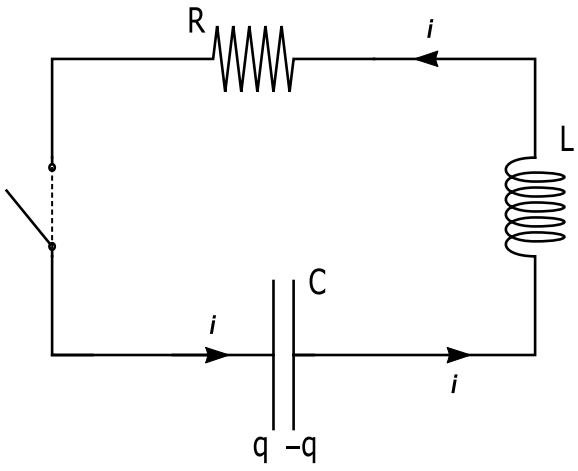


Fig. 9.25 Similar to Fig. 9.24 except for the battery which is missing here

To determine the current $i(t)$, differentiate (9.117).

$$i(t) = \frac{\mathrm{d}q(t)}{\mathrm{d}t} \; = \; \sigma_1 \, \varpi_1 \exp(\varpi_1 \, t) + \sigma_2 \, \varpi_2 \exp(\varpi_2 \, t). \tag{9.123}$$

Because the current $i(0)$ is vanishing, (9.123) gives

$$i(0) = \; \sigma_1 \, \varpi_1 + \sigma_2 \, \varpi_2 \; = \; 0 \; A \; . \tag{9.124}$$

Combining (9.122) and (9.124) leads to

$$\begin{aligned} \sigma_1 &= -\left(\frac{\varpi_2}{\varpi_1 - \varpi_2}\right) q(0) = -\left(\frac{\varpi_2}{\varpi_1 - \varpi_2}\right) 10^{-3} \; . \; C \\ \sigma_2 &= \left(\frac{\varpi_1}{\varpi_1 - \varpi_2}\right) q(0) = \left(\frac{\varpi_1}{\varpi_1 - \varpi_2}\right) 10^{-3} . C \end{aligned} \tag{9.125}$$

According to (9.114) and (9.116), one needs $\left(\frac{R}{2\,L}\right)$ and $\left(\frac{1}{L\,C}\right)$ for calculating α, ϖ_1, and ϖ_2. Using the numbers provided, one gets

$$\begin{aligned} \left(\frac{R}{2\,L}\right) = 500 \; ; \quad \left(\frac{1}{L\,C}\right) \; = \; 90,000 \; ; \quad \alpha \; = \; 400 \; ; \\ \varpi_1 = -100 \; ; \quad \varpi_2 \; = \; -900 \, ; \\ \sigma_1 = 9\left(\frac{10^{-3}}{8}\right) \; ; \quad \sigma_2 \; = \; -\left(\frac{10^{-3}}{8}\right) \; . \end{aligned} \tag{9.126}$$

Therefore, (9.117) and (9.123) give

$$\begin{aligned} q(t) &= \left(\frac{10^{-3}}{8}\right) \left[9 \exp(-100\,t) - \exp(-900\,t)\right] \; C \; ; \\ i(t) &= -\left(\frac{9}{80}\right) \left[\exp(-100\,t) - \exp(-900\,t)\right] \; A \; . \end{aligned} \tag{9.127}$$

Charge Reduction Exponential

The magnitude of the charge $q(t)$ on the plates reduces exponentially as time passes. The behavior of the current, however, is more interesting (Fig. 9.26).

Current Decrease Exponential

Although it stays negative, it goes through an extremum at $t = 0.0025993$ s when the current has reached -0.0759057 A. Thereafter the current strength decreases exponentially (Fig. 9.27).

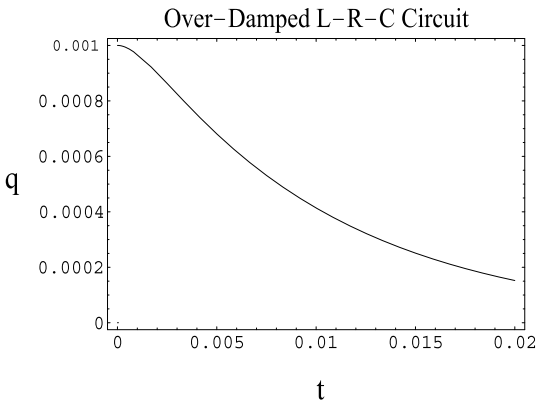


Fig. 9.26 An over-damped L-R-C series circuit: the capacitor charge $q(t)$ C is plotted as function of time t S

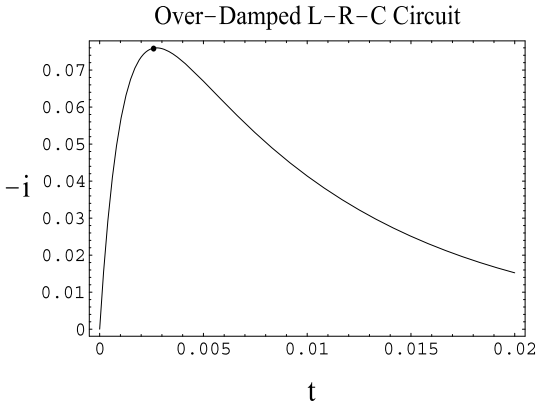


Fig. 9.27 Current $-i(t)$ in an over-damped L-R-C series circuit as function of time t. The extremum in $-i$ is 0.0759 A and it occurs when $t = 0.0026$ S

9.5.5 Critically Damped Series Circuit

Capacitor Charge $q(t)$

If the discriminant—see (9.118)—α^2 is vanishing, meaning if

$$\left(\frac{R}{2L}\right)^2 - \left(\frac{1}{LC}\right) = 0\,, \tag{9.128}$$

the two roots of the characteristic equation are equal. That is,

$$\varpi_1 = \varpi_2 \equiv \varpi_0 = -\left(\frac{R}{2L}\right). \tag{9.129}$$

Therefore, according to the well-established procedure—see (3.37)—(9.117) does not apply. Rather, the charge $q(t)$ must then be expressed in the form

$$q(t) = (\sigma_3 + \sigma_4\, t)\, \exp(\varpi_0\, t) \quad . \tag{9.130}$$

As a result, the current $i(t)$ is

$$i(t) = \frac{\mathrm{d}q}{\mathrm{d}t} = [\varpi_0\ (\sigma_3 + \sigma_4\, t) + \sigma_4]\, \exp(\varpi_0\, t) \quad . \tag{9.131}$$

9.5.6 Examples Group V

Problem

Assume the capacitor charge is $q(0)$ as the impressed voltage is turned off at $t = 0$. Immediately thereafter, the capacitor begins discharging . Work out the time dependence of the electric current and the (magnitude) of the charge on a capacitor plate. For numerical calculation use :

$$\begin{aligned} &Inductance\ L\ =\ 1\ H\ ;\ \ Resistance\ R\ =\ 2 \times 10^2\ \ \Omega\ ; \\ &Capacitance\ C\ =\ \left(10^4\right)^{-1}\ F\ ;\ \ Capacitor\ Charge\ \ q(0)\ =\ 10^{-4}\ C. \end{aligned}$$

These numbers obey (9.128) which is a requirement for a critically damped circuit.

Solution

At $t = 0$, the charge is $q(0)$ and the current $i(0)$. Inserting this information into (9.130) and (9.131), as well as noting the fact that $i(0) = 0$, we are led to

$$\begin{aligned} q(0) &= \sigma_3\ , \\ i(0)\ &=\ 0 = \omega_0\ \sigma_3 + \sigma_4 \end{aligned} \tag{9.132}$$

which gives

$$-\,\omega_0\, q(0) = \sigma_4 \quad . \tag{9.133}$$

Using the results obtained in (9.132) and (9.133) in (9.130) and (9.131), one gets

$$q(t)\ =\ q(0)\, \left[1 + \left(\frac{R}{2\,L}\right) t\right] \exp\left(-\frac{R\,t}{2\,L}\right) \tag{9.134}$$

and

$$i(t)\ =\ -\,q(0)\, t\, \left(\frac{R}{2\,L}\right)^2 \exp\left[-\left(\frac{R}{2\,L}\right) t\right]. \tag{9.135}$$

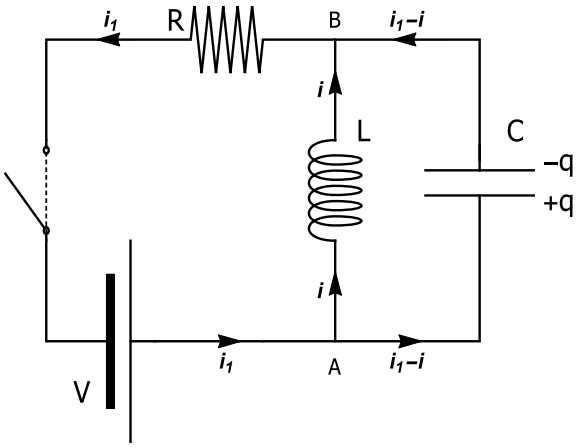


Fig. 9.28 L-R-C circuit with battery. Current $i_1(t)$ flows out and on arrival at point A current $i(t)$ separates and moves up the inductor L. The remaining current continues, eventually joining with $i(t)$ at point B

Turn the switch on in the $L-R-C$ circuit shown in Fig. 9.28. The large battery supplies current at constant voltage V volts. At time t, current $i_1(t)$ is flowing out of the positive terminal of the battery. At the first circuit point A, current $i(t)$ separates from $i_1(t)$ and moves up the inductor L to the circuit point B. The remaining current— that is, $i_1(t) - i(t)$—continues onward and moves up the capacitor C eventually joining at the second circuit point B with the current $i(t)$. Together these two currents make up current $i_1(t)$ which, after passing through the resistor R, returns to the negative terminal of the battery: thereby completing the closed-loop circuit. Assume at time t, the charge on the lower plate of the capacitor is $+q(t)$ and that on the higher plate is $-q(t)$.

9.5.7 Examples Group VI

Problem

Refer to Fig. 9.28 and work out currents $i(t)$, $i_1(t)$ and charge $q(t)$.

Solution

Figure 9.28 actually describes three separate circuits. Each follows its own differential equation and makes its own contribution to the process.

For instance, the first circuit on the left that contains the battery, the inductor L, and the resistor R. Its differential equation is

$$V - L\,\frac{\mathrm{d}i(t)}{\mathrm{d}t} - R\,i_1(t) \;=\; 0\,. \tag{9.136}$$

Next, there is the L-C circuit that is contained within points A and B. As shown in Fig. 9.24, both the inductor L and the capacitor C experience identical potential difference that obtains between points A and B: hence the equality

$$L\,\frac{\mathrm{d}i(t)}{\mathrm{d}t} \;=\; \frac{q(t)}{C} \tag{9.137}$$

where

$$\frac{\mathrm{d}q(t)}{\mathrm{d}t} \;=\; i_1(t) - i(t)\;. \tag{9.138}$$

In addition to the above three equations, one can also write a fourth equation that refers to the circuit heading from the battery through the capacitor back to the battery. That is,

$$V \;=\; \frac{q(t)}{C} + R\,i_1\;. \tag{9.139}$$

It turns out that (9.139) does not provide any new information beyond what is already included in the first set of three equations (9.136), (9.137), and (9.138). But that is not a problem because the first set of equations is sufficient for determining the three unknowns: $i(t)$, $i_1(t)$ and q(t). In order to carry out this determination, proceed as follows.

Differentiate (9.137) and combine the result with (9.138).

$$\begin{aligned} L\,\frac{\mathrm{d}^2 i(t)}{\mathrm{d}t^2} &= \frac{1}{C}\,\frac{\mathrm{d}q(t)}{\mathrm{d}t} \\ &= \frac{i_1(t) - i(t)}{C}\;. \end{aligned} \tag{9.140}$$

Eliminate $i_1(t)$ from (9.140) and (9.136),

$$L\,\frac{\mathrm{d}^2 i(t)}{\mathrm{d}t^2} + \frac{L}{R\,C}\,\frac{\mathrm{d}i(t)}{\mathrm{d}t} + \frac{i(t)}{C} \;=\; \left(\frac{V}{R\,C}\right)\,, \tag{9.141}$$

and divide (9.141) by L. This leads to the desired, single, differential equation whereby $i(t)$ can be determined.

$$\frac{\mathrm{d}^2 i(t)}{\mathrm{d}t^2} + \left(\frac{1}{R\,C}\right)\,\frac{\mathrm{d}i(t)}{\mathrm{d}t} + \left(\frac{1}{L\,C}\right)\,i(t) \;=\; \left(\frac{V}{L\,R\,C}\right)\;. \tag{9.142}$$

The above is an inhomogeneous linear ordinary differential equation with constant coefficients. Its solution, according to (3.59), is

$$i(t) \;=\; S_{\mathrm{comp}}(t) + I_{pi}(t) \tag{9.143}$$

where $S_{\text{comp}}(t)$ is the complementary solution and $I_{pi}(t)$ is the particular integral that are defined as follows.

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \left(\frac{1}{R\,C}\right)\frac{\mathrm{d}}{\mathrm{d}t} + \left(\frac{1}{L\,C}\right)\right] S_{\text{comp}}(t) \;=\; 0\;. \tag{9.144}$$

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \left(\frac{1}{R\,C}\right)\frac{\mathrm{d}}{\mathrm{d}t} + \left(\frac{1}{L\,C}\right)\right] I_{pi}(t) \;=\; \left(\frac{V}{L\,R\,C}\right)\;. \tag{9.145}$$

According to the description given in detail in Chap. 3—see (3.55) and (3.56)—$S_{\text{comp}}(t)$ is readily calculated. Similarly, $I_{pi}(t)$ may be calculated by using (3.61) and (3.67).

$$\begin{aligned} S_{\text{comp}}(t) \;=\; & \sigma_1 \exp\left(-\frac{t}{2\,R\,C}\right)\exp\left\{t\sqrt{\left(\frac{1}{2\,R\,C}\right)^2 - \left(\frac{1}{L\,C}\right)}\right\} \\ & + \sigma_2 \exp\left(-\frac{t}{2\,R\,C}\right)\exp\left\{-t\sqrt{\left(\frac{1}{2\,R\,C}\right)^2 - \left(\frac{1}{L\,C}\right)}\right\}. \end{aligned} \tag{9.146}$$

$$I_{pi}(t) \;=\; \left(\frac{V}{L\,R\,C}\right)\frac{1}{\left(\frac{1}{L\,C}\right)} \;=\; \left(\frac{V}{R}\right)\;. \tag{9.147}$$

Following (9.137) → (9.147), both the charge q(t) and the current $i_1(t)$ can now be calculated. For instance, (9.137) gives

$$q(t) \;=\; L\,C\frac{\mathrm{d}i(t)}{\mathrm{d}t} \tag{9.148}$$

which leads to

$$\begin{aligned} q(t) = & \left(\frac{-L}{2R}\right)\sigma_1 \exp\left(-\frac{t}{2\,R\,C}\right)\exp\left\{t\sqrt{\left(\frac{1}{2\,R\,C}\right)^2 - \left(\frac{1}{L\,C}\right)}\right\} \\ & + \left(\frac{-L}{2R}\right)\sigma_2 \exp\left(-\frac{t}{2\,R\,C}\right)\exp\left\{-t\sqrt{\left(\frac{1}{2\,R\,C}\right)^2 - \left(\frac{1}{L\,C}\right)}\right\} \\ & + \sqrt{\left(\frac{L}{2R}\right)^2 - (LC)}\left[\sigma_1 \exp\left(-\frac{t}{2\,R\,C}\right)\exp\left\{t\sqrt{\left(\frac{1}{2\,R\,C}\right)^2 - \left(\frac{1}{L\,C}\right)}\right\}\right] \\ & - \sqrt{\left(\frac{L}{2R}\right)^2 - (LC)}\left[\sigma_2 \exp\left(-\frac{t}{2\,R\,C}\right)\exp\left\{-t\sqrt{\left(\frac{1}{2\,R\,C}\right)^2 - \left(\frac{1}{L\,C}\right)}\right\}\right]. \end{aligned} \tag{9.149}$$

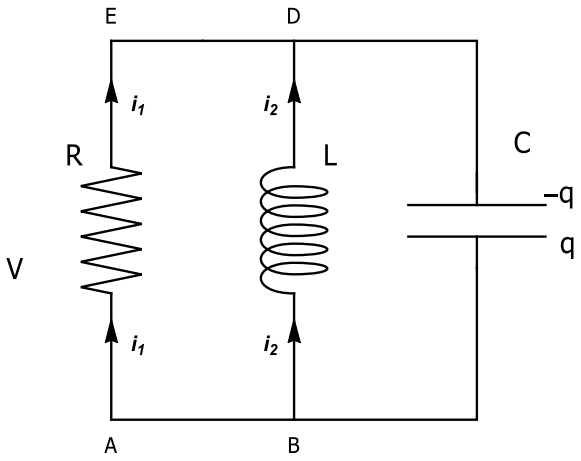


Fig. 9.29 At time 0, the two plates of the capacitor hold charges Q_0 and $-Q_0$. Shown here is q(t) versus $i_1(t)$, and $i_2(t)$

And knowing $q(t)$ and $i(t)$—see (9.157), (9.143), (9.146), and (9.147)—(9.138) provides a simple route to the evaluation of the current $i_1(t)$.

9.5.8 Examples Group VII

Problem

At time $t = 0$, the two plates of the capacitor hold charges $q(0) \equiv Q_0$ and $-Q_0$ and current $i_2(t = 0)$ is equal to I_{20}. Work out $q(t), i_1(t)$, and $i_2(t)$. The relevant plot is in Fig. 9.29.

Solution

The potential difference across points A and E is the same as that across points B and D.

$$R\, i_1(t) \;=\; L\,\frac{\mathrm{d}i_2(t)}{\mathrm{d}t}\;. \tag{9.150}$$

The potential difference across points B and D is also equal to that across the capacitor C.

$$L\,\frac{\mathrm{d}i_2(t)}{\mathrm{d}t} \;=\; q(t)/C \quad . \tag{9.151}$$

Additionally, the current flow out of the positive plate—and back into the negative plate—of the capacitor leads to the relationship

$$\frac{\mathrm{d}q(t)}{\mathrm{d}t} \;=\; -\,i_1(t) - i_2(t) \;. \tag{9.152}$$

Differentiate (9.151) and combine the result with (9.152).

$$\begin{aligned} L\,\frac{\mathrm{d}i^2{}_2(t)}{\mathrm{d}t^2} &= \frac{1}{C}\,\frac{\mathrm{d}q(t)}{\mathrm{d}t} \\ &= \frac{-i_1(t) - i_2(t)}{C} \;. \end{aligned} \tag{9.153}$$

Now eliminate $i_1(t)$ from (9.153) and (9.150) and divide the result by L to get

$$\frac{\mathrm{d}^2 i_2(t)}{\mathrm{d}t^2} + \frac{1}{R\,C}\,\frac{\mathrm{d}i_2(t)}{\mathrm{d}t} + \frac{i_2(t)}{L\,C} \;=\; 0 \;. \tag{9.154}$$

Much like (9.146), differential equation (9.154) leads to

$$\begin{aligned} i_2(t) \;=\; &\sigma_1 \exp\left(-\frac{t}{2\,R\,C}\right) \exp\left\{t\sqrt{\left(\frac{1}{2\,R\,C}\right)^2 - \left(\frac{1}{L\,C}\right)}\right\} \\ &+ \sigma_2 \exp\left(-\frac{t}{2\,R\,C}\right) \exp\left\{-t\sqrt{\left(\frac{1}{2\,R\,C}\right)^2 - \left(\frac{1}{L\,C}\right)}\right\} \;. \end{aligned} \tag{9.155}$$

According to (9.150) and (9.151)

$$\begin{aligned} i_1(t) &= \left(\frac{L}{R}\right)\frac{\mathrm{d}i_2(t)}{\mathrm{d}t} \\ &= \left(\frac{1}{R\,C}\right) q(t) \;. \end{aligned} \tag{9.156}$$

The relationship $L\,C\,\frac{\mathrm{d}i_2(t)}{\mathrm{d}t} \;\equiv\; q(t)$ leads to

$$\begin{aligned} q(t) = &\left(\frac{-\,L}{2R}\right)\sigma_1 \exp\left(-\frac{t}{2\,R\,C}\right) \exp\left\{t\sqrt{\left(\frac{1}{2\,R\,C}\right)^2 - \left(\frac{1}{L\,C}\right)}\right\} \\ &+ \left(\frac{-\,L}{2R}\right)\sigma_2 \exp\left(-\frac{t}{2\,R\,C}\right) \exp\left\{-t\sqrt{\left(\frac{1}{2\,R\,C}\right)^2 - \left(\frac{1}{L\,C}\right)}\right\} \\ &+ \sqrt{\left(\frac{L}{2R}\right)^2 - (LC)}\left[\sigma_1 \exp\left(-\frac{t}{2\,R\,C}\right) \exp\left\{t\sqrt{\left(\frac{1}{2\,R\,C}\right)^2 - \left(\frac{1}{L\,C}\right)}\right\}\right] \end{aligned}$$

$$-\sqrt{\left(\frac{L}{2R}\right)^2-(LC)}\left[\sigma_2\exp\left(-\frac{t}{2\,R\,C}\right)\exp\left\{-t\sqrt{\left(\frac{1}{2\,R\,C}\right)^2-\left(\frac{1}{L\,C}\right)}\right\}\right].$$

$$\equiv i_1(t)\;RC\;. \tag{9.157}$$

Equations (9.155) and (9.157) provide the solution to Examples Group 7 except that there still are two unknowns σ_1 and σ_2. The given two boundary conditions, namely $i_2(t=0)=I_{20}$ and q(t = 0) = Q_0, determine these unknowns. That is,

$$I_{20}=\sigma_1+\sigma_2 \tag{9.158}$$

and

$$Q_0=-\left(\frac{-L}{2R}\right)(\sigma_1+\sigma_2)+(\sigma_1-\sigma_2)\sqrt{\left(\frac{L}{2R}\right)^2-(LC)}\;. \tag{9.159}$$

Chapter 10
Numerical Solution



Given a first-order linear differential equation

$$\frac{\mathrm{d}y(x)}{\mathrm{d}x} = F(x, y) \quad ,$$

and its solution, $y(x_0)$, at a point $x = x_0$, Runge–Kutta procedure is used to estimate $y(x_0 + \Delta)$. Runge–Kutta procedure is the least accurate when it uses only one step for the entire move. And indeed, as noted later, the single-step process does yield grossly inaccurate results. The two-step process—see (10.12) and (10.13)—improves the results only slightly. But the four-step effort—see (10.14)–(10.17)—does much better. It reduces the error to about $\left(\frac{1}{50}\right)$th of that for the one-step process. Estimates from a ten-step Runge–Kutta process are recorded in Table 10.1. These estimates—being in error only by $\left(100 \times \frac{0.0025}{22.17}\right) = 0.0113\%$—are highly accurate.

Coupled first-order differential equations are treated next.

$$\begin{aligned} A(x, y) &= \frac{\mathrm{d}x}{\mathrm{d}t} = x + y \quad , \\ B(x, y) &= \frac{\mathrm{d}y}{\mathrm{d}t} = x - y \quad . \end{aligned} \tag{10.1}$$

Together these equations are equivalent to a single second-order differential equation. Tables 10.2, 10.3, 10.4, 10.5, 10.6, and 10.7 display numerical results for the one-step, two-step, and the five-step processes. Table 10.8 contains numerical results gathered during a twenty-step Runge–Kutta process. At maximum extension, $\Delta = 2$, the Runge–Kutta estimate is 26.371190. It differs from the exact result, 26.371404, by only a tiny amount, 0.000214. The percentage error involved is 0.000811. Table 10.9 records numerical results collected during a twenty-step Runge–Kutta process. At maximum extension, $\Delta = 2$, the Runge–Kutta estimate for Y_n is 11.0170347. It differs from the exact result, 11.0171222, by 0.0000875. The percentage error involved is 0.000794. It is similar to the corresponding error, 0.000811%, for X_n.

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The accuracy achieved by the twenty-step Runge–Kutta estimate is quite extraordinary. When very high accuracy is desired, the twenty-step Runge–Kutta process yields results that are worth the effort.

10.1 Single First-Order Differential Equations

10.1.1 Runge–Kutta Steps

Consider a single first-order differential equation.

$$\frac{\mathrm{d}y(x)}{\mathrm{d}x} = F(x, y) \ . \tag{10.2}$$

Its solution, $y(x)$, contains one arbitrary constant that can be determined by specifying one boundary condition. Let that boundary condition be the value of $y(x)$ at $x = x_0$. That is

$$y(x_0) \ = \ Y_0 \ . \tag{10.3}$$

Assume the given differential equation cannot be solved by methods that have been described so far in this book. The objective of the current exercise is to work out an approximation procedure whereby one can proceed beyond the starting point in a step-by-step process. Each step is chosen to be of length ϵ. In this manner, n steps are needed to move from the initial location x_0 to the desired final location x_n.

$$x_n \ = \ x_0 + n\,\epsilon \ = \ x_0 + \Delta \ . \tag{10.4}$$

In other words, knowing $y_0 = y(x_0) = Y_0$, one attempts to find Y_n that is an approximation to $y_n = y(x_n) = y(x_0 + \Delta)$. However, before starting, one must choose numerical values for the distance Δ and the total number of steps, n. As a result one is led to the single-step length ϵ specified by

$$\epsilon = \frac{\Delta}{n} \ . \tag{10.5}$$

The protocol for carrying out this objective is suggested by Runge and Kutta. It involves successive use of small 'steps,' each of which carries the process through a small extension in position, say from x to $(x+\epsilon)$. To the above purpose, one proceeds as follows. The first Runge–Kutta step of length ϵ takes us from the starting point x_0 to a neighboring point $x_1 = x_0 + \epsilon$ and requires calculating the parameters $R_{1;1}, R_{1;2}, R_{1;3}, R_{1;4}, K_{1;1}$. Here Y_1 represents an estimate of the exact result $y_1 = y(x_1) = y(x_0 + \epsilon)$. Of course, the hope is that the extension, ϵ, is short enough that the estimate Y_1 is a good approximation to the unknown exact result y_1.

Choose the value of x_0, $Y_0 = y(x_0)$, n, and Δ. Set $\epsilon = \frac{\Delta}{n}$.

Runge–Kutta : First Step

$$\begin{aligned}
&\epsilon = \ldots \ ; \ x_0 = \ldots \ ; \ Y_0 = \ldots \ ; \\
&R_{1;1} = \epsilon\, F\,(x_0, Y_0) \ ; \ R_{2;1} = \epsilon\, F\left(x_0 + \frac{\epsilon}{2}, Y_0 + \frac{R_{1;1}}{2}\right) ; \\
&R_{3;1} = \epsilon\, F\left(x_0 + \frac{\epsilon}{2}, Y_0 + \frac{R_{2;1}}{2}\right) \ ; \ R_{4;1} = \epsilon\, F\left(x_0 + \epsilon, Y_0 + R_{3;1}\right) \ ; \\
&K_1 = \frac{1}{6}\left(R_{1;1} + 2\,R_{2;1} + 2\,R_{3;1} + R_{4;1}\right) \ ; \\
&Y_1 = Y_0 + K_1 \ . \qquad (10.6)
\end{aligned}$$

It is important to note that Y_0 has been chosen to be exactly equal to a given value. Therefore, it is exact. On the other hand, Y_1 is the Runge–Kutta estimated value of y_1. Therefore, it is approximate. The objective of the current exercise is to make this approximation as accurate as needed. And an important option for achieving this is to make ϵ short.

Because ϵ is short, one needs to extend through several ϵ's. Usually, the beginning and the endpoints, say x_0 and $x_0 + \Delta$, are determined by the physical needs of the subject matter that is being studied. And, of course, greater the number n, the greater the effort needed to arrive at the desired estimate. Therefore, the choice for n is made by the amount of effort one is willing to expand. And because of the relationship (10.5), that decision fixes the value of ϵ.

We could now proceed to the second step. But it is more convenient directly to go to the nth step.

Runge–Kutta : *n*th-Step

$$\begin{aligned}
&\epsilon = \ldots \ ; \ x_{n-1} = \ldots \ ; \ Y_{n-1} = \ldots \ ; \\
&R_{1;n} = \epsilon\, F\,(x_{n-1}, Y_{n-1}) \ ; \ R_{2;n} = \epsilon\, F\left(x_{n-1} + \frac{\epsilon}{2}, Y_{n-1} + \frac{R_{1;n}}{2}\right) \\
&R_{3;n} = \epsilon\, F\left(x_{n-1} + \frac{\epsilon}{2}, Y_{n-1} + \frac{R_{2;n}}{2}\right) \ ; \ R_{4;n} = \epsilon\, F\left(x_{n-1} + \epsilon, Y_{n-1} + R_{3;n}\right) \\
&K_n = \frac{1}{6}\left(R_{1;n} + 2\,R_{2;n} + 2\,R_{3;n} + R_{4;n}\right) \ ; \\
&Y_n = Y_{n-1} + K_n \ . \qquad (10.7)
\end{aligned}$$

10.1.2 Runge–Kutta Solution

The Runge–Kutta procedure is best demonstrated by working out an actual problem. To that purpose, consider the following first-order differential equation.

$$\frac{\mathrm{d}y(x)}{\mathrm{d}x} \;=\; x\,y \;. \tag{10.8}$$

Assume we do not know how to solve it. Because it is first order, its solution must contain one arbitrary constant that can be determined by a single boundary condition. Let such boundary condition be its solution at $x = x_0 = 0$ and assume that this solution is equal to $Y_0 = 3$. That is

$$y(x = x_0 = 0) \;=\; Y_0 \;= 3 \;. \tag{10.9}$$

The Objective

Knowing x_0 and $y(x_0)$, we wish to find the solution $y(x)$ at a point $x = x_0 + \Delta$.

Need for Exact Solution

As mentioned earlier, in principle we do not know the exact solution of (10.8). For educational purposes, however, it is important to know how well the Runge–Kutta procedure actually is faring. To that end, it is best to have available, hidden somewhere in the background, the exact solution. Then, as we proceed through the various Runge–Kutta steps, the accuracy of the numerical approximation can properly be evaluated.

The exact boundary-value solution of (10.8) is

$$y(x) = a_0\,\exp\left(\frac{x^2}{2}\right) \;=\; 3\,\exp\left(\frac{x^2}{2}\right) \;. \tag{10.10}$$

Solution with Longest Possible ϵ

In order to make our work the shortest possible, we need to use the longest-possible ϵ : meaning $\epsilon = \Delta = 2$. Then, working through the first step described in (10.6) we have

$$\begin{aligned}
\epsilon = 2 \;;\;\; x_0 \;&=\; 0 \;;\;\; Y_0 \;=\; 3 \;; \\
R_{1;1} = \epsilon\, x_0\, Y_0 \;&=\; 0 \;;\; R_{2;1} \;=\; \epsilon\left(x_0 + \frac{\epsilon}{2}\right)\left(Y_0 + \frac{R_{1;1}}{2}\right) \;=\; 6 \;; \\
R_{3;1} &= \epsilon\left(x_0 + \frac{\epsilon}{2}\right)\left(Y_0 + \frac{R_{2;1}}{2}\right) \;=\; 12 \;; \\
R_{4;1} &= \epsilon\,(x_0 + \epsilon)\left(Y_0 + R_{3;1}\right) \;=\; 60 \;; \\
K_1 &= \frac{1}{6}\left(R_{1;1} + 2\,R_{2;1} + 2\,R_{3;1} + R_{4;1}\right) \;=\; 16 \;; \\
Y_1 &= Y_0 + K_1 \;=\; 3 + 16 = \;19 \;.
\end{aligned} \tag{10.11}$$

The Y_1 calculated above is the estimated value of $y(x = \Delta)$. Its exact value is $y(x = 2) = 3\exp(\frac{2^2}{2}) = 22.167168$. Thus, for the longest-possible choice for ϵ, the

Runge–Kutta estimate is quite inaccurate. It equals 19. Therefore, it is in error by 3.17 points which is about $\left(100 \times \frac{3.17}{22.17}\right) = 14.3\%$.

Solution When ϵ Is Half-the-Longest Possible

Next, let us deal with the case where ϵ is equal to half-the-longest possible.

Given ϵ is equal to one-half the maximum-possible length—meaning $\epsilon = \frac{\Delta}{2} = 1$—one needs to walk through two successive Runge–Kutta steps. For the first step, one has [Note: Compare (10.6)]

$$\begin{aligned} \epsilon = 1 \ , \ x_0 = 0 \ , \ Y_0 = 3 \ ; \\ R_{1;1} = 0 \ ; \ R_{2;1} = 1.5 \ ; \\ R_{3;1} = 1.875 \ ; \ R_{4;1} = 4.875 \\ K_1 = \frac{1}{6}\left(R_{1;1} + 2\,R_{2;1} + 2\,R_{3;1} + R_{4;1}\right) = 1.9375 \ ; \\ Y_1 = Y_0 + K_1 = 3 + 1.9375 = 4.9375 \, . \end{aligned} \tag{10.12}$$

And for the second step

$$\begin{aligned} \epsilon = 1 \ ; \ x_1 = \epsilon \ ; \ Y_1 = 4.9375 \ ; \\ R_{1;2} = e\,x_1\,Y_1 = 4.9375 \ ; \ R_{2;2} = 11.1094 \ ; \\ R_{3;2} = 15.7383 \ ; \ R_{4;2} = 41.3516 \ ; \\ K_2 = \frac{1}{6}\left(R_{1;2} + 2\,R_{2;2} + 2\,R_{3;2} + R_{4;2}\right) = 16.6641 \ ; \\ Y_2 = Y_1 + K_2 = 4.9375 + 16.6641 = 21.6016 \, . \end{aligned} \tag{10.13}$$

$Y_2 = 21.6016$, calculated above, is the estimated value of $y(2)$. Its exact value is $3\exp(2) = 22.167168$. Thus, for half-the-longest-possible choice for ϵ, the Runge–Kutta estimate is already much improved over that for the longest-possible choice. The result is now only 0.57 points in error, which is less than a fifth of the previous error.

Solution with One-Fourth the Longest Possible ϵ

Next, let us consider an even shorter value of ϵ equal to a quarter of the longest-possible choice: meaning $\epsilon = 0.5$. This requires trudging through four Runge–Kutta steps, each of length ϵ. The first step yields

$$\begin{aligned} \epsilon = 0.5 \ ; \ x_0 = 0 \ ; \ Y_0 = 3 \ ; \\ R_{1;1} = 0 \ ; \ R_{2;1} = 0.375 \ ; \\ R_{3;1} = 0.398438 \ ; \ R_{4;1} = 0.849608 \ ; \\ K_1 = \frac{1}{6}\left(R_{1;1} + 2\,R_{2;1} + 2\,R_{3;1} + R_{4;1}\right) = 0.399414 \ ; \\ Y_1 = Y_0 + K_1 = 3 + 0.399414 = 3.399414. \end{aligned} \tag{10.14}$$

The second step gives

$$\begin{aligned}
\epsilon &= 0.5 \;;\; x_1 = \epsilon \;;\; Y_1 = 3.399414 \;; \\
R_{1;2} &= 0.849853 \;;\; R_{2;2} = 1.43413 \;; \\
R_{3;2} &= 1.54368 \;;\; R_{4;2} = 2.47154 \;; \\
K_2 &= \frac{1}{6}\left(R_{1;2} + 2\,R_{2;2} + 2\,R_{3;2} + R_{4;2}\right) = 1.54617 \;; \\
Y_2 &= Y_1 + K_2 = 3.399414 + 1.54617 = 4.945584.
\end{aligned} \tag{10.15}$$

The third step gives

$$\begin{aligned}
\epsilon &= 0.5 \;;\; x_2 = 2\epsilon \;;\; Y_2 = 4.945584 \;; \\
R_{1;3} &= 2.47279 \;;\; R_{2;3} = 3.86373 \;; \\
R_{3;3} &= 4.2984 \;;\; R_{4;3} = 6.93299 \;; \\
K_3 &= \frac{1}{6}\left(R_{1;3} + 2\,R_{2;3} + 2\,R_{3;3} + R_{4;3}\right) = 4.28834 \;; \\
Y_3 &= Y_2 + K_3 = 4.945584 + 4.28834 = 9.233924
\end{aligned} \tag{10.16}$$

The fourth- and the final step leads to

$$\begin{aligned}
\epsilon &= 0.5 \;;\; x_3 = 3\epsilon \;;\; Y_3 = 9.233924 \;; \\
R_{1;4} &= 6.92544 \;;\; R_{2;4} = 11.1096 \;; \\
R_{3;4} &= 12.9401 \;;\; R_{4;4} = 12.174 \;; \\
K_4 &= \frac{1}{6}\left(R_{1;2} + 2\,R_{2;2} + 2\,R_{3;2} + R_{4;2}\right) = 12.8665 \;; \\
Y_4 &= Y_3 + K_4 = 9.233924 + 12.8665 = 22.100424
\end{aligned} \tag{10.17}$$

Equation (10.17) indicates that the use of one-quarter the longest-possible ϵ leads to Runge–Kutta approximation result, 22.100424. This result is in error by a mere $22.167 - 22.100 = 0.067$ points. Being only $\left(100 \times \frac{0.067}{22.167}\right) = 0.302\%$, this is greatly reduced error. Indeed, it is almost one-fiftieth of $\left(100 \times \frac{22.167-19}{22.167}\right) = 14.29\%$—which was the error when the longest-possible ϵ was used.

Solution with One-Tenth the Longest-Possible ϵ

To get a good feel as to whether the improvement in the estimate is worth the additional effort, let us try even a shorter length ϵ : say, $\epsilon = \frac{\Delta}{10} = 0.2$. In order to save space, these results are tabulated in Table 10.1.

Table 10.1 indicates that when the travel to the maximum extension, $x_{10} = 2 = \Delta$, is carried out by using $\epsilon = \frac{\Delta}{10}$—meaning, when ten steps are used to get to the maximum extension— the Runge–Kutta estimate is in error ≈ 0.0025. The percentage error now—being about 0.0113%—is absolutely minuscule. Indeed it is much less than one-thousandth the error, 14.3%, found by using the longest possible value of ϵ.

Table 10.1 Ten-step numerical approximation results

x_n	Exact	Runge–Kutta	Error
0.	3	3	0
0.2	3.060604	3.060604	0.0
0.4	3.249861	3.249861	0.0
0.6	3.591652	3.591651	0.000001
0.8	4.131383	4.131378	0.000005
1.0	4.946164	4.946149	0.000015
1.2	6.163260	6.163255	0.00005
1.4	7.993369	7.993240	0.00013
1.6	10.789919	10.789563	0.00036
1.8	15.159271	15.158317	0.00096
2.0	22.167168	22.164669	0.0025

Summing-Up: Results for One-Tenth the Longest-Possible ϵ

'For an initial value first-order differential equation, Runge–Kutta provides excellent approximation to the exact result when ten Runge–Kutta steps are used. Indeed, using less than half this effort—meaning when one-fourth the longest-possible ϵ is used–the resultant Runge–Kutta estimate is in error by only 0.302% which may often be acceptable accuracy.

10.2 Coupled Differential Equations First Order

Second-order differential equations can be constructed from coupled first-order differential equations. For instance, consider the following differential equations:

$$\begin{aligned} A(x, y) &= \frac{\mathrm{d}x}{\mathrm{d}t} = x + y \quad , \\ B(x, y) &= \frac{\mathrm{d}y}{\mathrm{d}t} = x - y \quad . \end{aligned} \tag{10.18}$$

Differentiate with respect to t the upper (10.18) and write the result as follows.

$$\begin{aligned} \frac{d^2x}{\mathrm{d}t^2} &= \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t} + x - y = \frac{\mathrm{d}x}{\mathrm{d}t} + x + x - \frac{\mathrm{d}x}{\mathrm{d}t} \\ &= 2\,x \ . \end{aligned} \tag{10.19}$$

Similarly, differentiate the lower (10.18) and write

$$\begin{aligned} \frac{d^2y}{\mathrm{d}t^2} &= \frac{\mathrm{d}x}{\mathrm{d}t} - \frac{\mathrm{d}y}{\mathrm{d}t} = x + y - \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}t} + y + y - \frac{\mathrm{d}y}{\mathrm{d}t} \\ &= 2\,y \ . \end{aligned} \tag{10.20}$$

The second-order differential equations (10.19) and (10.20) can straightforwardly be solved by the usual, trial solution as exponentials, technique. Meaning set $x(t) = \text{const.}\exp(\alpha\, t)$ and $y(t) = \text{const.}\exp(\beta\, t)$. In this manner, the solutions are

$$\begin{aligned} x(t) &= a_1 \exp\left(\sqrt{2}\, t\right) + b_1 \exp\left(-\sqrt{2}\, t\right) \ , \\ y(t) &= a_2 \exp\left(\sqrt{2}\, t\right) + b_2 \exp\left(-\sqrt{2}\, t\right) \ . \end{aligned} \tag{10.21}$$

While we expect the solution to a first-order differential equation to contain only a single arbitrary constant, unfortunately the given solutions—see (10.21)— to two first-order differential equations seem to involve a total of four such constants. These constants are $a_1,\ b_1,\ a_2,\ b_2$. Fortunately, by using the procedure described in (3.28)–(3.37), one can properly reduce these constants to the needed total of only two. The result is

$$\begin{aligned} x(t) &= a_1\ \exp\left(\sqrt{2}\, t\right) + b_1\ \exp\left(-\sqrt{2}\, t\right) \ , \\ y(t) &= \left\{\sqrt{2}-1\right\}\ a_1\ \exp\left(\sqrt{2}\, t\right) - \left\{\sqrt{2}+1\right\}\ b_1\ \exp\left(-\sqrt{2}\, t\right) \ . \end{aligned} \tag{10.22}$$

And the two constants a_1 and b_1 can be determined by the two boundary conditions—namely

$$\begin{aligned} x(t = t_0 = 0) &= 1 \\ &= a_1 + b_1 \ = \ X_0 \ ; \\ y(t = t_0 = 0) &= 2 \\ &= \left\{\sqrt{2}-1\right\}\ a_1 - \left\{\sqrt{2}+1\right\}\ b_1 \ = \ Y_0 \ . \end{aligned} \tag{10.23}$$

Equation (10.23) lead to

$$a_1 \ = \ \left(\frac{3+\sqrt{2}}{2\sqrt{2}}\right) \quad ; \quad b_1 \ = \ \left(\frac{\sqrt{2}-3}{2\sqrt{2}}\right) \ . \tag{10.24}$$

Inserting the a_1 and b_1—given in (10.24)—into (10.22) provides the desired exact solution of (10.18).

$$\begin{aligned} x(t) &= \left(\frac{3+\sqrt{2}}{2\sqrt{2}}\right) \exp\left(\sqrt{2}\, t\right) + \left(\frac{\sqrt{2}-3}{2\sqrt{2}}\right)\ \exp\left(-\sqrt{2}\, t\right) \ ; \\ y(t) &= \left\{\sqrt{2}-1\right\}\ \left(\frac{3+\sqrt{2}}{2\sqrt{2}}\right) \exp\left(\sqrt{2}\, t\right) \\ &\quad - \left\{\sqrt{2}+1\right\}\ \left(\frac{\sqrt{2}-3}{2\sqrt{2}}\right) \exp\left(-\sqrt{2}\, t\right) \ . \end{aligned} \tag{10.25}$$

Equation (10.25) duly satisfy the required two boundary conditions (10.23). And they provide ready access to the exact result for comparison with Runge–Kutta estimate.

One hastens to add that the Runge–Kutta numerical approximation—to be discussed in the following—for a coupled pair of first-order differential equations can be employed without knowing the exact results in (10.25). The latter are provided merely for educational purposes. In other words, exact results help evaluate the accuracy of the Runge–Kutta estimate.

10.2.1 Runge–Kutta Steps

Use the information $\epsilon = 0.1,\ t_0 = 0,\ x(t_0) = X_0 = 1,\ y(t_0) = Y_0 = 2$ and note the following:

X_n is the estimated value of $x_n = x(t_n) = x(t_0 + n\,\epsilon)$, Y_n is the estimated value of $y_n = y(t_n) = y(t_0 + n\,\epsilon)$, and t_0 has been set equal to zero.

The two coupled first-order differential equations that need to be solved—subject to the two boundary conditions (10.23)—are A(x,y) and B(x,y) as given in (10.18).

The nth Runge–Kutta step refers to $t_n = t_0 + n\,\epsilon$ with $t_0 = 0$ and x_n and y_n are exact results.

10.2.2 Numerical Solution

One-Step Numerical Approximation
Solution with the Longest-Possible ϵ

Let us start with the worst possible choice for ϵ : meaning ϵ equal to the farthest separation envisioned. That is $\epsilon = \Delta = 2$. Remember, $X_0 = 1$ and $Y_0 = 2$ and the Runge–Kutta objective is to move to $X_0 + \Delta = 1 + 2 = 3$ and $Y_0 + \Delta = 2 + 2 = 4$. The results are provided in Tables 10.2 and 10.3. In the single-step process, the Runge–Kutta estimate for final X_n is in error by 4.7. The exact value is 26.37. In percentage terms, for the one-step process, the error in final X_n is $100\left(\frac{4.7}{26.37}\right) = 17.8\%$. The corresponding error in final value of Y_n is $100\left(\frac{0.35}{11.02}\right) = 3.18\%$.

Table 10.2 $\epsilon = 2$: one-step approximation for X_n

$n\,\epsilon$	Exact	Runge–Kutta	Error
0	1.	1.	0.
2	26.371404	21.66667	4.7

Table 10.3 $\epsilon = 2$: one-step approximation for Y_n

$n\,\epsilon$	Exact	Runge–Kutta	Error
0	2.	2.	0.
2	11.017122	10.666667	0.35

Relationships and Results for Runge–Kutta First Step

The objective at the first step is to calculate X_1 and Y_1 which are estimates for the corresponding exact values $x_1 = x(t_1) = x(t_0 + \epsilon)$ and $y_1 = y(t_1) = y(t_0 + \epsilon)$. Note $t_0 = 0$. The Runge–Kutta equation and the relevant numerical values for the first step are given below.

$$
\begin{aligned}
R_{1;1} &= \epsilon\ (X_0 + Y_0) \ = \ 0.1 \times (1 + 2) \ = \ 0.3\ ; \\
S_{1;1} &= \epsilon\ (X_0 - Y_0) \ = \ 0.1 \times (1 - 2) \ = \ -0.1\ ; \\
R_{2;1} &= \epsilon \left(X_0 + \frac{R_{1;1}}{2} + Y_0 + \frac{S_{1;1}}{2} \right) \ = \ 0.1 \left[1 + \frac{0.3}{2} + 2 - \frac{0.1}{2} \right] = 0.31 \\
S_{2;1} &= \epsilon \left(X_0 + \frac{R_{1;1}}{2} - Y_0 - \frac{S_{1;1}}{2} \right) \ = \ 0.1 \left[1 + \frac{0.3}{2} - 2 + \frac{0.1}{2} \right] = -0.08 \\
R_{3;1} &= \epsilon \left(X_0 + \frac{R_{2;1}}{2} + Y_0 + \frac{S_{2;1}}{2} \right) \ = \ 0.1 \left[1 + \frac{0.31}{2} + 2 - \frac{0.08}{2} \right] = 0.3115 \\
S_{3;1} &= \epsilon \left(X_0 + \frac{R_{2;1}}{2} - Y_0 - \frac{S_{2;1}}{2} \right) \ = \ 0.1 \left[1 + \frac{0.31}{2} - 2 + \frac{0.08}{2} \right] = -0.0805 \\
R_{4;1} &= \epsilon \left(X_0 + R_{3;1} + Y_0 + S_{3;1} \right) \ = \ 0.1\,[1 + 0.3115 + 2 - 0.0805] = 0.3231 \\
S_{4;1} &= \epsilon \left(X_0 + R_{3;1} - Y_0 - S_{3;1} \right) \ = \ 0.1\,[1 + 0.3115 - 2 + 0.0805] = -0.0608 \\
K_1 &= \frac{1}{6} \left(R_{1;1} + 2\,R_{2;1} + 2\,R_{3;1} + R_{4;1} \right) \ = \ 0.311017; \\
L_1 &= \frac{1}{6} \left(S_{1;1} + 2\,S_{2;1} + 2\,S_{3;1} + S_{4;1} \right) \ = \ -0.0803; \\
X_1 &= X_0 + K_1 \ = \ 1.311017 \ ; \ \ Y_1 \ = \ Y_0 + L_1 \ = \ 1.919700\ ; \\
x_1 &= 1.311018 \ ; \ \ y_1 \ = \ 1.919700\ .
\end{aligned}
\tag{10.26}
$$

Detailed information about the second step is provided next.

Relationships and Results for Runge–Kutta Second Step

$$
\begin{aligned}
R_{1;2} &= \epsilon\ (X_1 + Y_1) \ = \ 0.1 \times (1.311018 + 1.9197) \ = \ 0.323072\ ; \\
S_{1;2} &= \epsilon\ (X_1 - Y_1) \ = \ 0.1 \times (1.311018 - 1.9197) \ = \ -0.060868\ ; \\
R_{2;2} &= \epsilon \left(X_1 + \frac{R_{1;2}}{2} + Y_1 + \frac{S_{1;2}}{2} \right) \ = \ 0.336182 \\
S_{2;2} &= \epsilon \left(X_1 + \frac{R_{1;2}}{2} - Y_1 - \frac{S_{1;2}}{2} \right) \ = \ -0.0416713 \\
R_{3;2} &= \epsilon \left(X_1 + \frac{R_{2;2}}{2} + Y_1 + \frac{S_{2;2}}{2} \right) \ = \ 0.337797 \\
S_{3;2} &= \epsilon \left(X_1 + \frac{R_{2;2}}{2} - Y_1 - \frac{S_{2;2}}{2} \right) \ = \ -0.0419756 \\
R_{4;2} &= \epsilon \left(X_1 + R_{3;2} + Y_1 + S_{3;2} \right) \ = \ 0.352654
\end{aligned}
$$

$$
\begin{aligned}
S_{4;2} &= \epsilon\,\left(X_1 + R_{3;2} - Y_1 - S_{3;2}\right) \;=\; -0.022891 \\
K_2 &= \frac{1}{6}\left(R_{1;2} + 2\,R_{2;2} + 2\,R_{3;2} + R_{4;2}\right) \;=\; 0.337281; \\
L_2 &= \frac{1}{6}\left(S_{1;2} + 2\,S_{2;2} + 2\,S_{3;2} + S_{4;2}\right) \;=\; -0.041842; \\
X_2 &= X_1 + K_2 \;=\; 1.648298; \quad Y_2 \;=\; Y_1 + L_2 \;=\; 1.877858\,; \\
x_2 &= 1.648299 \;\;;\;\; y_1 \;=\; 1.877857\,.
\end{aligned}
\tag{10.27}
$$

The second step refers to $t_2 = t_0 + 2\epsilon \;= 0.2$. X_2 is the estimated value of x_2. And Y_2 is the estimated value of y_2. Note, t_0 has been set equal to zero.

If the desired total number of Runge–Kutta steps is n, one must proceed step-by step starting at the first step. The structure of the nth Runge–Kutta step is the following.

Expected Structure of Runge–Kutta *n*th Step

$$
\begin{aligned}
R_{1;n} &= \epsilon\;(X_{n-1} + Y_{n-1})\;; \\
S_{1;n} &= \epsilon\;(X_{n-1} - Y_{n-1})\;; \\
R_{2;n} &= \epsilon\,\left(X_{n-1} + \frac{R_{1;n}}{2} + Y_{n-1} + \frac{S_{1;n}}{2}\right) \\
S_{2;n} &= \epsilon\,\left(X_{n-1} + \frac{R_{1;n}}{2} - Y_{n-1} - \frac{S_{1;n}}{2}\right) \\
R_{3;n} &= \epsilon\,\left(X_{n-1} + \frac{R_{2;n}}{2} + Y_{n-1} + \frac{S_{2;n}}{2}\right) \\
S_{3;n} &= \epsilon\,\left(X_{n-1} + \frac{R_{2;n}}{2} - Y_{n-1} - \frac{S_{2;n}}{2}\right) \\
R_{4;n} &= \epsilon\,\left(X_{n-1} + R_{3;n} + Y_{n-1} + S_{3;n}\right) \\
S_{4;n} &= \epsilon\,\left(X_{n-1} + R_{3;n} - Y_{n-1} - S_{3;n}\right) \\
K_n &= \frac{1}{6}\left(R_{1;n} + 2\,R_{2;n} + 2\,R_{3;n} + R_{4;n}\right) \\
L_n &= \frac{1}{6}\left(S_{1;n} + 2\,S_{2;n} + 2\,S_{3;n} + S_{4;n}\right) \\
X_n &= X_{n-1} + K_n\;;\;\; Y_n \;=\; Y_{n-1} + L_n \\
x_n \;&=\; \;-\ldots\;;\;\; y_n \;=\; \ldots
\end{aligned}
\tag{10.28}
$$

Two-Step Numerical Approximation
Solution with Half-the-Longest-Possible ϵ

Consider next the case $\epsilon = 1 = \frac{\Delta}{2}$. Keep $X_0 = 1,\; Y_0 = 2,\; X_0 + \Delta = X_0 + 2\epsilon = 1 + 2 = 3$ and $Y_0 + \Delta = Y_0 + 2\epsilon = 2 + 2 = 4$. The results for X_n and Y_n are recorded in Tables 10.4 and 10.5. Here the Runge–Kutta estimate for final X_n is

Table 10.4 $\epsilon = 1$: two-step approximation for X_n

$n\,\epsilon$	Exact	Runge–Kutta	Error
0	1.	1.	0.
1	6.28308	6.166667	0.1164
2	26.371404	25.583335	0.7881

Table 10.5 $\epsilon = 1$: two-step approximation for Y_n

$n\,\epsilon$	Exact	Runge–Kutta	Error
0	2.	2.	0.
1	3.	3.0	0.
2	11.0171222	10.722223	0.2949

Table 10.6 $\epsilon = 0.4$: five-step approximation for X_n

$n\,\epsilon$	Exact	Runge–Kutta	Error
0	2.	2.	0.
0.4	2.429344	2.428267	0.001
0.8	4.657032	4.653827	0.003
1.2	8.415135	8.407175	0.008
1.6	14.938662	14.920358	0.020
2	26.371404	26.331396	0.040

Table 10.7 $\epsilon = 0.4$: five-step approximation for Y_n

$n\,\epsilon$	Exact	Runge–Kutta	Error
0	2.	2.	0.
0.4	1.906948	1.9072	0.0002
0.8	2.440566	2.440032	0.0005
1.2	3.776214	3.773594	0.0026
1.6	6.352821	6.345752	0.0071
2	11.017122	11.000915	0.016207

in error by 0.7881. The exact value is 26.37. In percentage terms, for the two-step Runge–Kutta process, the error in final value of X_n is $100\left(\frac{0.7881}{26.37}\right) = 2.9884\%$. That is about one-sixth the error, 17.8, for the single-step process.

Regarding the final Y_n, the relevant Tables 10.3 and 10.5. The single-step estimate for Y_n is in error by 0.3505 and the two-step processes are in error by 0.2949. In other words, while the error involved in the single-step process is 3.18%, the two-step process is in error by 2.68%. The difference between the two sets of errors is not significant. It is only 0.5%. Perhaps things will improve if we go to a five-step process.

Five-Step Numerical Approximation
Solution with One-Fifth the Longest-Possible ϵ
Error Comparison : One-Step Versus Five-Step Process

Examine Table 10.6. At full extension Δ the Runge–Kutta estimate for X_n is 26.331396. Compared with the exact result, 26.371404, the estimate is in error by 0.04008. This error is about $(\frac{1}{120})$th the error, 4.705, found by using the one-step Runge–Kutta process. In percentage terms, the error here is approximately 0.152%.

Table 10.8 $\epsilon = 0.1$: twenty-step approximation for X_n

$n\,\epsilon$	Exact	Runge–Kutta	Error
0	1.	1.	0.
0.1	1.311018	1.311017	0.000001
0.2	1.648299	1.648298	0.000001
0.3	2.0186022	2.018599	0.000003
0.4	2.429344	2.429340	0.000004
0.5	2.888754	2.888748	0.000006
0.6	3.406036	3.406027	0.000011
0.7	3.991552	3.991540	0.000012
0.8	4.657032	4.657016	0.000016
0.9	5.415807	5.415787	0.000020
1.0	6.283080	6.283054	0.000026
1.1	7.276224	7.276192	0.000033
1.2	8.415135	8.415094	0.000041
1.3	9.722630	9.722578	0.000052
1.4	11.224901	11.224837	0.000064
1.5	12.952045	12.951966	0.000079
1.6	14.938662	14.938565.	0.000097
1.7	17.22455	17.224431	0.000112
1.8	19.855504	19.855359	0.000128
1.9	22.884230	22.884054	0.000147
2.0	26.371404	26.371190	0.000214

Consider Y_n next. The improvement in percentage error is quite dramatic. According to Table 10.7, at full extension the exact result, 11.017122, and the Runge–Kutta estimate, 11.000915, differ by 0.016207. As such the error in the Runge–Kutta estimate after five steps is $\frac{1}{\left(\frac{0.3505}{0.016207}\right)} = \left(\frac{1}{21.63}\right)$th that for the one-step process. In percentage terms, the error in Y_n, after five Runge–Kutta steps, is $\left(\frac{0.016207}{11.00092}\right) \times 100 = 0.147$. As such, after five Runge–Kutta steps, the percentage error in Y_n is about the same as that in X_n.

Considering the limited amount of effort involved in working through only five Runge–Kutta steps, the resultant error of only about 0.15%—in estimates for both X_n and Y_n—is reasonable. However, if one needs to pursue much greater accuracy, one might try a twenty-step Runge–Kutta process.

Twenty-Step Numerical Approximation Solution with One-Twentieth the Longest-Possible ϵ

Refer to Table 10.8 for numerical results for X_n gathered during a twenty-step Runge–Kutta process. At maximum extension, $\Delta = 2$, the Runge–Kutta estimate for X_n is 26.371190. It differs from the exact result, 26.371404, by only a tiny amount, 0.000214. The percentage error involved is 0.000811.

Table 10.9 records numerical results Y_n collected during a twenty-step Runge–Kutta process. At maximum extension, $\Delta = 2$, the Runge–Kutta estimate for Y_n is

Table 10.9 $\epsilon = 0.1$: twenty-step approximation for Y_n

$n\,\epsilon$	Exact	Runge–Kutta	Error
0	2.	2.	0.000000
0.1	1.919700	1.919700	0.000000
0.2	1.877857	1.877858	0.000001
0.3	1.873635	1.873635	0.000001
0.4	1.906948	1.906947	0.000001
0.5	1.978463	1.978462	0.000001
0.6	2.089613	2.089612	0.000001
0.7	2.242626	2.242623	0.000003
0.8	2.440566	2.440561	0.000005
0.9	2.687398	2.687392	0.000006
1.0	2.988068	2.988059	0.000009
1.1	3.348599	3.3485877	0.000011
1.2	3.776214	3.776199	0.000015
1.3	4.279479	4.279460	0.000019
1.4	4.868477	4.868452	0.000025
1.5	5.555006	5.554975	0.000031
1.6	6.352821	6.352782	0.000039
1.7	7.277904	7.277856	0.000048
1.8	8.348788	8.348729	0.000059
1.9	9.586926	9.586854	0.000072
2.0	11.0171222	11.0170347	0.0000875

11.0170347. It differs from the exact result, 11.0171222, by 0.0000875. The percentage error involved is 0.000794. It is similar to the corresponding error, 0.000811%, for X_n. The accuracy achieved by the twenty-step Runge–Kutta estimate is quite extraordinary. When high accuracy is essential, the twenty-step process is well worth the additional effort.

Chapter 11
Frobenius Solution



11.1 Normalized Form

For $A(x) \neq 0$, the normalized form of differential equation

$$A(x)\,y''(x) + B(x)\,y'(x) + C(x)\,y(x) \;=\; 0 \tag{11.1}$$

is

$$y''(x) + M(x)\,y'(x) + N(x)\,y(x) \;=\; 0\;, \tag{11.2}$$

where

$$M(x) \;=\; \frac{B(x)}{A(x)}\;;\;\; N(x) \;=\; \frac{C(x)}{A(x)}\;. \tag{11.3}$$

11.1.1 An Analytic Function

A function $F(x)$ is analytic at $x = x_0$ if the Taylor series,

$$F(x) \;=\; \sum_{n=0}^{\infty} (x - x_0)^n \,\frac{\left(\frac{\mathrm{d}^n F}{\mathrm{d}x^n}\right)}{n!}\;, \tag{11.4}$$

converges to $F(x)$ for all x at, and in the neighborhood of, x_0. Note that geometric functions, polynomials, and exponentials of analytic functions are analytic as are rational functions except at the points where their denominator is vanishing.

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11.1.2 Ordinary Point

If both coefficients, $M(x)$ and $N(x)$, are analytic at $x = x_0$ the point x_0 is called an 'Ordinary Point' of the differential equation (11.2).

11.1.3 Regular Singular Point

On the other hand, if either or both of these coefficients is/are not analytic at $x = x_0$, but both $(x - x_0) M(x)$ and $(x - x_0)^2 N(x)$ are analytic at $x = x_0$, then x_0 is a 'Regular Singular Point' of the differential equation (11.2).

11.1.4 Irregular Singular Point

But if one and/or other of the products $(x - x_0) M(x)$ and $(x - x_0)^2 N(x)$ is/are not analytic at $x = x_0$, the x_0 is an 'Irregular Singular Point' of the given differential equation.

11.1.5 Solution Around Ordinary Point

All solutions—compare, Whittaker, E. T. and *Watson*[30]—of a given differential equation, say (11.2), at or immediately around its ordinary point, say $x = x_0$, are analytic and may be expressed as Taylor series such as (11.4). Note, the radius of convergence of these solutions is equal to at least the separation between x_0 and the nearest singular point of the given differential equation.

11.1.6 Equations of Type (a)

In the following, we shall treat differential equations of type (a)—see (11.5)—around their ordinary point $x = 0$ with initial conditions $u(x = 0) = \sigma_1$ and $\left(\frac{du}{dx}\right)_{x=0} = \sigma_2$.

$$(a) \quad : \quad \left[D^2 + \alpha\, x\, D + (\beta\, x^2 + \gamma)\right] u(x) \;=\; 0 \;, \tag{11.5}$$

11.1.7 Solution

In (11.5), the parameters α, β, γ are constants that are real. We classify these equations as being of type (a). Clearly $x = 0$ is an ordinary point of the differential equation (11.5). [Note: The symbols $M(x)$ and $N(x)$ are defined in (11.2)]. Because both the relevant coefficients $M(x) = \alpha x$ and $N(x) = \left(\beta x^2 + \gamma\right)$ are analytic at $x = 0$. Frobenius solution of a differential equation around its ordinary point, $x = 0$, is found in simple form. Simple form refers to the case where the unknown constant ν_0 is set qual to 0, and the Frobenius solution is expressed as a *Maclaurin*[27] series.

$$u(x) \;=\; \sum_{n=0}^{\infty} a_n\, x^n \;. \tag{11.6}$$

In order to use (11.6), the differentials that would be needed in (11.5) are the following:

$$\begin{aligned} D^2u(x) \;&=\; \sum_{n=0}^{\infty} n(n-1)\, a_n\, x^{n-2} \;=\; \sum_{n=2}^{\infty} n(n-1)\, a_n\, x^{n-2} \;; \\ Du(x) \;&=\; \sum_{n=0}^{\infty} n\, a_n\, x^{n-1} \;=\; \sum_{n=1}^{\infty} n\, a_n\, x^{n-1} \;. \end{aligned} \tag{11.7}$$

Accordingly, the Frobenius expansion of differential equation (11.5) is

$$\sum_{n=2}^{\infty} n(n-1)\, a_n\, x^{n-2} + \alpha x \sum_{n=1}^{\infty} n\, a_n\, x^{n-1} + (\beta x^2 + \gamma) \sum_{n=0}^{\infty} a_n\, x^n \;=\; 0\,. \tag{11.8}$$

Unfortunately (11.8), in its current form, is not convenient for working out the details of the unknown parameters a_n. To relieve this inconvenience, one would want to achieve two things. First: All infinite sums should have the same range. Second: All infinite sums should post the same power of x, say x^n.

To begin this process consider, the first term on the left-hand side in (11.8) and manipulate it as follows:

$$\begin{aligned} \sum_{n=2}^{\infty} n(n-1)\, a_n\, x^{n-2} \;&=\; 2\, a_2 + 6\, a_3\, x + \sum_{n=4}^{\infty} n(n-1)\, a_n\, x^{n-2} \\ &=\; 2\, a_2 + 6\, a_3\, x + \sum_{n=2}^{\infty} (n+2)(n+1)\, a_{n+2}\, x^n \;. \end{aligned} \tag{11.9}$$

[Note: In order to render the term $\sum_{n=4}^{\infty} n(n-1)\, a_n\, x^{n-2}$ into $\sum_{n=2}^{\infty} (n+2)(n+1)\, a_{n+2}\, x^n$, set $n - 2 = n'$ so that $n = n' + 2$. Then write

$\sum_{n=4}^{\infty} n(n-1)\,a_n\,x^{n-2}$ as $\sum_{n'+2=4}^{\infty}(n'+2)(n'+1)\,a_{n'+2}\,x^{n'}$ and finally change the notation $n' \to n$ to get the desired result.]

Next look at the second term on the left-hand side in (11.8) and manipulate it so that the infinite sum, like (11.9), is summed from n = 2 and has x^n in it. Fortunately that is easy to achieve.

$$\alpha\,x\sum_{n=1}^{\infty} n\,a_n\,x^{n-1} \;=\; \alpha\,a_1\,x + \alpha\sum_{n=2}^{\infty} n\,a_n\,x^n. \tag{11.10}$$

Finally consider the last term on the left-hand side in (11.8). One gets

$$\sum_{n=0}^{\infty} a_n\,x^{(n+2)} \;=\; \sum_{n'=2}^{\infty} a_{n'-2}\,x^{n'}. \tag{11.11}$$

And another change of variable—this time from $n' \to n$—does the trick.

$$\begin{aligned}(\beta\,x^2+\gamma)\sum_{n=0}^{\infty} a_n\,x^n \;&=\; \gamma\sum_{n=0}^{\infty} a_n\,x^n + \beta\sum_{n=0}^{\infty} a_n\,x^{(n+2)}\\ &=\; \gamma\,a_0 + \gamma\,a_1\,x + \gamma\sum_{n=2}^{\infty} a_n\,x^n + \beta\sum_{n=2}^{\infty} a_{n-2}\,x^n\;.\end{aligned} \tag{11.12}$$

[Note: To see how $\beta\sum_{n=0}^{\infty} a_n\,x^{(n+2)}$ becomes $\beta\sum_{n=2}^{\infty} a_{n-2}\,x^n$, set $n' = n+2$ and change variable from $n \to n'$.]

The original differential equation (11.8) is written as: [(11.9)] + [(11.10)] + [(11.12)] = 0. That is

$$\begin{aligned}&[2\,a_2 + 6\,a_3\,x + \sum_{n=2}^{\infty}(n+2)(n+1)\,a_{n+2}\,x^n] + [\alpha\,a_1\,x + \alpha\sum_{n=2}^{\infty} n\,a_n\,x^n]\\ &\quad+[\gamma\,a_0 + \gamma\,a_1\,x + \gamma\sum_{n=2}^{\infty} a_n\,x^n + \beta\sum_{n=2}^{\infty} a_{n-2}\,x^n] \;=\; 0\;.\end{aligned} \tag{11.13}$$

Or more conveniently as

$$\begin{aligned}&(\gamma\,a_0 + 2\,a_2) + x(\alpha\,a_1 + \gamma\,a_1 + 6\,a_3)\\ &+\sum_{n=2}^{\infty} x^n\{(n+2)(n+1)\,a_{n+2} \;+\; (\alpha\,n+\gamma)\,a_n + \beta\,a_{n-2}\} = \;0\;.\end{aligned} \tag{11.14}$$

Equation (11.14) should be true for all values of x. And that can happen only if the coefficient multiplying any given power of x is zero: meaning, only if the following relationships hold:

$$(\gamma\, a_0 + 2\, a_2) \;=\; 0 \;\;,\;\; (\alpha\, a_1 + \gamma\, a_1 + 6\, a_3) \;=\; 0 \;\;, \tag{11.15}$$

and for $n \geq 2$,

$$(n+2)(n+1)\, a_{n+2} \;+\; a_n(\alpha\, n + \gamma) + \beta\, a_{n-2} = \; 0 \quad . \tag{11.16}$$

Equations (11.15) and (11.16) lead to

$$a_2 \;=\; -\left(\frac{\gamma}{2}\right) a_0 \quad ; \quad a_3 \;=\; -\left(\frac{\alpha+\gamma}{6}\right) a_1 \;\;, \tag{11.17}$$

and for all $n \geq 2$

$$a_{n+2} \;=\; -\left[\frac{a_n(\alpha\, n + \gamma) + \beta\, a_{n-2}}{(n+2)(n+1)}\right] \;\;. \tag{11.18}$$

Before proceeding further, it is important to recognize that the differential equation being solved here is second order and it should have two arbitrary constants for which a_0 and a_1 would clearly serve well. Constants a_2 and a_3 are already known in terms of these two. All other constants, namely a_4 and higher, have to be extracted from (11.18) and represented in terms of a_0 and a_1. That requires beginning with $n = 2$ in (11.18).

$$a_{2+2} \;=\; -\left[\frac{(2\,\alpha+\gamma)\, a_2 + \beta\, a_0}{12}\right] \;=\; \left[\frac{(2\,\alpha+\gamma)\left(\frac{\gamma}{2}\right) - \beta}{12}\right] a_0 \;\;. \tag{11.19}$$

Next use $n = 3$ in (11.18).

$$a_{2+3} \;=\; -\left[\frac{(3\,\alpha+\gamma)\, a_3 + \beta\, a_1}{20}\right] \;=\; \left[\frac{(3\,\alpha+\gamma)\,\left(\frac{\alpha+\gamma}{6}\right) - \beta}{20}\right] a_1 \;\;. \tag{11.20}$$

With $n = 4$ (11.18) gives

$$a_6 = -\left[\frac{(4\,\alpha+\gamma)\, a_4 + \beta\, a_2}{30}\right] = -\left[\frac{(4\,\alpha+\gamma)\,\left(\frac{(2\,\alpha+\gamma)\left(\frac{\gamma}{2}\right)-\beta}{12}\right) - \beta\,\left(\frac{\gamma}{2}\right)}{30}\right] a_0 \;\;. \tag{11.21}$$

$$a_7 = -\left[\frac{(5\alpha+\gamma)\, a_5 + \beta\, a_3}{42}\right] = -\left[\frac{(5\alpha+\gamma)\left[\frac{(3\alpha+\gamma)\left(\frac{\alpha+\gamma}{6}\right)-\beta}{20}\right] - \beta\,\left(\frac{\alpha+\gamma}{6}\right)}{42}\right] a_1 \tag{11.22}$$

Finally, the initial conditions must be satisfied. That is

$$u(x=0) \;=\; \sigma_1 \;=\; a_0 \quad ; \quad \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)_{x=0} \;=\; \sigma_2 \;=\; a_1 . \tag{11.23}$$

Inserting (11.17) and (11.19)–(11.23) into (11.6) leads to the result—see (11.24) given below—for the simple form of the Frobenius series solution of (11.5).

$$\begin{aligned} u(x) = & \left\{ 1 - \left[\frac{\gamma}{2}\right] x^2 + \left[\frac{(2\,\alpha+\gamma)\left(\frac{\gamma}{2}\right)-\beta}{12}\right] x^4 \right\} \sigma_1 \\ & - \left\{ \left[\frac{(4\,\alpha+\gamma)\left(\frac{(2\,\alpha+\gamma)\left(\frac{\gamma}{2}\right)-\beta}{12}\right) - \beta\left(\frac{\gamma}{2}\right)}{30}\right] x^6 \right\} \sigma_1 \;+\; O(x^8)\,\sigma_1 \\ & + \left\{ x - \left[\frac{\alpha+\gamma}{6}\right] x^3 + \left[\frac{(3\,\alpha+\gamma)\left(\frac{\alpha+\gamma}{6}\right)-\beta}{20}\right] x^5 \right\} \sigma_2 \\ & - \left\{ \left[\frac{(5\,\alpha+\gamma)\left[\frac{(3\,\alpha+\gamma)\left(\frac{\alpha+\gamma}{6}\right)-\beta}{20}\right] - \beta\left(\frac{\alpha+\gamma}{6}\right)}{42}\right] x^7 \right\} \sigma_2 \;+ O(x^9)\,\sigma_2 \;. \end{aligned} \tag{11.24}$$

One good thing about having arbitrary constants, α, β, γ, in (11.5) is that different choices of these constants lead to different differential equations. And its solution in the form of (11.24) makes the first several terms of their Frobenius series solution instantly available.

11.1.8 Examples Group I

To see how the above process works, let us solve the following five differential equations which are similar in form to (11.5) which is of type (a).

$$\begin{aligned} I-(A) \;&:\; [D^2 + 2\,x\,D + (3\,x^2+4)]u(x) \;=\; 0 \;, \\ I-(B) \;&:\; [D^2 + x\,D + (2\,x^2+1)]u(x) \;=\; 0 \;, \\ I-(C) \;&:\; [D^2 + 2\,x\,D + (x^2-1)]u(x) \;=\; 0 \;. \\ I-(D) \;&:\; [D^2 + 3\,x\,D + (2\,x^2+1)]u(x) \;=\; 0 \;, \\ I-(E) \;&:\; [D^2 - x\,D + (-x^2-1)]u(x) \;=\; 0 \;. \end{aligned} \tag{11.25}$$

All we have to do to solve the (11.25) is to use the values of α, β, and γ that are given. First few terms of the relevant Frobenius power series solution of equations I-(A)–I-(E) are the following.

For differential equation I-(A), use: $\alpha = 2, \beta = 3, \gamma = 4$. Therefore, its solution:

$$I-(A)\ :\ u(x) = \sigma_1\left[1-2x^2+\left(\frac{13}{12}\right)x^4-\left(\frac{7}{30}\right)x^6+O(x^8)\right] + \sigma_2\left[x-x^3+\left(\frac{7}{20}\right)x^5-\left(\frac{19}{420}\right)x^7+O(x^9)\right]. \quad (11.26)$$

For I-(B) use: $\alpha = 1, \beta = 2, \gamma = 1$, therefore

$$I-(B)\ :\ u(x) = \sigma_1\left[1-\left(\frac{1}{2}\right)x^2-\left(\frac{1}{24}\right)x^4+\left(\frac{29}{720}\right)x^6+O(x^8)\right] + \sigma_2\left[x-\left(\frac{1}{3}\right)x^3-\left(\frac{1}{30}\right)x^5+\left(\frac{13}{630}\right)x^7+O(x^9)\right]. \quad (11.27)$$

For I-(C) use: $\alpha = 2, \beta = 1, \gamma = -1$, therefore

$$I-(C)\ :\ u(x) = \sigma_1\left[1+\left(\frac{1}{2}\right)x^2-\left(\frac{5}{24}\right)x^4+\left(\frac{23}{720}\right)x^6+O(x^8)\right] + \sigma_2\left[x-\left(\frac{1}{6}\right)x^3-\left(\frac{1}{120}\right)x^5+\left(\frac{29}{5040}\right)x^7+O(x^9)\right]. \quad (11.28)$$

For I-(D) use: $\alpha = 3, \beta = 2, \gamma = 1$, therefore

$$I-(D)\ :\ u(x) = \sigma_1\left[1-\left(\frac{1}{2}\right)x^2+\left(\frac{1}{8}\right)x^4-\left(\frac{1}{48}\right)x^6+O(x^8)\right] + \sigma_2\left[x-\left(\frac{2}{3}\right)x^3+\left(\frac{7}{30}\right)x^5+-\left(\frac{2}{35}\right)x^7+O(x^9)\right]. \quad (11.29)$$

For I-(E) use: $\alpha = -1, \beta = -1, \gamma = -1$, therefore

$$I-(E)\ :\ u(x) = \sigma_1\left[1+\left(\frac{1}{2}\right)x^2+\left(\frac{5}{24}\right)x^4+\left(\frac{37}{720}\right)x^6+O(x^8)\right] + \sigma_2\left[x+\left(\frac{1}{3}\right)x^3+\left(\frac{7}{60}\right)x^5+\left(\frac{31}{1260}\right)x^7+O(x^9)\right]. \quad (11.30)$$

11.1.9 Problems Group I

$$
\begin{aligned}
I.1 &: [D^2 + x\,D + (-x^2 + 1)]u(x) = 0\ . \\
I.2 &: [D^2 + x\,D + (-3\,x^2 + 2)]u(x) = 0\ , \\
I.3 &: [D^2 + x\,D + (3\,x^2 - 2)]u(x) = 0\ , \\
I.4 &: [D^2 + 4\,x\,D + (-x^2 - 4)]u(x) = 0\ . \\
I.5 &: [D^2 + x\,D + (-x^2 + 1)]u(x) = 0\ . \\
I.6 &: [D^2 + x\,D + (x^2 + 1)]u(x) = 0\ , \\
I.7 &: [D^2 - x\,D - (x^2 + 1)]u(x) = 0\ , \\
I.8 &: [D^2 + 2\,x\,D + 2\,(x^2 + 1)]u(x) = 0\ . \\
I.9 &: [D^2 - 2\,x\,D - 2\,(x^2 + 1)]u(x) = 0\ . \\
I.10 &: [D^2 + 3\,x\,D + 3\,(x^2 - 1)]u(x) = 0\ .
\end{aligned}
\tag{11.31}
$$

11.1.10 Equations of Type (b)

Having solved Frobenius equations of Type (a) around their ordinary point, somewhat more complicated such equations—to be referred to as equations of Type (b)—are tackled below.

$$
(b) \ : \ \left[D^2 + (\alpha\,x^2 + \beta\,x + \gamma)\,D + (\mu\,x^2 + \nu\,x + \rho)\right]u(x) = 0\ . \tag{11.32}
$$

11.1.11 Solution

Again the initial conditions are: $u(x = 0) = \sigma_1$ and $\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)_{x=0} = \sigma_2$. All the six constants α, β, γ, μ, ν, ρ are real. Clearly $x = 0$ is an ordinary point of (11.32) because both the coefficients $M(x) = \left(\alpha\,x^2 + \beta\,x + \gamma\right)$ and $N(x) = \left(\mu\,x^2 + \nu\,x + \rho\right)$ are analytic at $x = 0$. Frobenius solution of a differential equation around its ordinary point $x = 0$ is best found in simple form. Simple form refers to the case where the unknown constant ν_0 is set qual to 0, and the Frobenius solution is expressed as a Maclaurin series.

$$
u(x) = \sum_{n=0}^{\infty} a_n\,x^n\ . \tag{11.33}
$$

Using (11.33), the relevant differentials for (11.32) (b) are:

$$D^2u(x) \;=\; \sum_{n=0}^{\infty} n(n-1)\,a_n\,x^{n-2} \;=\; \sum_{n=2}^{\infty} n(n-1)\,a_n\,x^{n-2}\;;$$
$$Du(x) \;=\; \sum_{n=0}^{\infty} n\,a_n\,x^{n-1} \;=\; \sum_{n=1}^{\infty} n\,a_n\,x^{n-1}\;. \tag{11.34}$$

Accordingly, the Frobenius version of differential equation (b) is

$$\begin{aligned}&\left[D^2 + (\alpha x^2 + \beta x + \gamma)\,D + (\mu x^2 + \nu x + \rho)\right] u(x)\\ &= \sum_{n=2}^{\infty} n(n-1)\,a_n\,x^{n-2} + (\alpha x^2 + \beta x + \gamma) \sum_{n=1}^{\infty} n\,a_n\,x^{n-1}\\ &+ (\mu x^2 + \nu x + \rho) \sum_{n=0}^{\infty} a_n\,x^n \;=\; 0\,. \end{aligned}\tag{11.35}$$

Unfortunately (11.35), in its current form, is not convenient for working out the details of the unknown parameters a_n. To relieve this inconvenience, one would want to achieve two things. First: All infinite sums should have the same range. Second: All infinite sums should post the same power of x, say x^n.

To begin this process, consider the first term on the right-hand side in (11.35) and manipulate it as follows:

$$\begin{aligned}\sum_{n=2}^{\infty} n(n-1)\,a_n\,x^{n-2} \;&=\; 2\,a_2 + 6\,a_3\,x + \sum_{n=4}^{\infty} n(n-1)\,a_n\,x^{n-2}\\ &=\; 2\,a_2 + 6\,a_3\,x + \sum_{n=2}^{\infty} (n+2)(n+1)\,a_{n+2}\,x^n\;.\end{aligned}$$

[Note: In order to render the term $\sum_{n=4}^{\infty} n(n-1)\,a_n\,x^{n-2}$ into $\sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}\,x^n$, set $n-2=n'$ so that $n=n'+2$. Then write $\sum_{n=4}^{\infty} n(n-1)\,a_n\,x^{n-2}$ as $\sum_{n'+2=4}^{\infty}(n'+2)(n'+1)\,a_{n'+2}\,x^{n'}$ and finally change the notation $n'\to n$ to achieve the given result $\sum_{n=2}^{\infty}(n+2)(n+1)\,a_{n+2}\,x^n$.]

Therefore, the first term on the right-hand side in (11.35) is

$$\sum_{n=2}^{\infty} n(n-1)\,a_n\,x^{n-2} \;=\; 2\,a_2 + 6\,a_3\,x + \sum_{n=2}^{\infty} (n+2)(n+1)\,a_{n+2}\,x^n\;. \tag{11.36}$$

Next manipulate the second term on the right-hand side in (11.35) and arrange it so that the sum incudes x^n and ranges from $n = 2$ to ∞.

$$(\alpha x^2 + \beta x + \gamma) \sum_{n=1}^{\infty} n a_n x^{n-1} = \alpha \sum_{n=2}^{\infty} (n-1) a_{n-1} x^n + \beta a_1 x + \beta \sum_{n=2}^{\infty} n a_n x^n + \gamma \sum_{n=1}^{\infty} n a_n x^{n-1} \quad . \tag{11.37}$$

The following transformations were used in (11.37). The first term on the left-hand side of (11.37) is

$$\alpha x^2 \sum_{n=1}^{\infty} n a_n x^{n-1} = \alpha \sum_{n=1}^{\infty} n a_n x^{n+1} \tag{11.38}$$

Set $n + 1 = n'$. As a result, the first term on the left-hand side of (11.37) becomes

$$\alpha x^2 \sum_{n=1}^{\infty} n a_n x^{n-1} = \alpha \sum_{n=1}^{\infty} n a_n x^{n+1} = \alpha \sum_{n'=2}^{\infty} (n'-1) a_{n'-1} x^{n'} = \alpha \sum_{n=2}^{\infty} (n-1) a_{n-1} x^n \quad . \tag{11.39}$$

The second term is $\beta x \sum_{n=1}^{\infty} n a_n x^{n-1}$. Write it as

$$\beta x \sum_{n=1}^{\infty} n a_n x^{n-1} = \beta \sum_{n=1}^{\infty} n a_n x^n = \beta a_1 x + \beta \sum_{n=2}^{\infty} n a_n x^n \quad . \tag{11.40}$$

The third term is $\gamma \sum_{n=1}^{\infty} n a_n x^{n-1}$. Set $n - 1 = n'$. As a result,

$$\gamma \sum_{n=1}^{\infty} n a_n x^{n-1} = \gamma \sum_{n'=0}^{\infty} (n'+1) a_{n'+1} x^{n'} = \gamma \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \gamma a_1 + 2\gamma a_2 x + \gamma \sum_{n=2}^{\infty} (n+1) a_{n+1} x^n \quad . \tag{11.41}$$

Consequently, the second term on the right-hand side in (11.35) is

$$
\begin{aligned}
&(\alpha x^2 + \beta x + \gamma) \sum_{n=1}^{\infty} n\, a_n\, x^{n-1} \\
&= \alpha \sum_{n=2}^{\infty} (n-1)\, a_{n-1}\, x^n + \beta\, a_1\, x + \beta \sum_{n=2}^{\infty} n\, a_n\, x^n \\
&+ \gamma\, a_1 + 2\,\gamma\, a_2\, x + \gamma \sum_{n=2}^{\infty} (n+1)\, a_{n+1}\, x^n \ . \qquad (11.42)
\end{aligned}
$$

Finally, consider the last term on the right-hand side in (11.35).

$$
\begin{aligned}
&(\mu x^2 + \nu x + \rho) \sum_{n=0}^{\infty} a_n\, x^n \\
&= \mu \sum_{n=0}^{\infty} a_n\, x^{n+2} + \nu \sum_{n=0}^{\infty} a_n\, x^{n+1} + \rho \sum_{n=0}^{\infty} a_n\, x^n \ . \qquad (11.43)
\end{aligned}
$$

By setting $n' = n + 2$ and changing variable from $n \to n'$ the sum $\sum_{n=0}^{\infty} a_n\, x^{n+2}$ becomes $\sum_{n'=2}^{\infty} a_{n'-2}\, x^{n'}$. Therefore,

$$
\mu \sum_{n=0}^{\infty} a_n\, x^{n+2} \ = \ \mu \sum_{n'=2}^{\infty} a_{n'-2}\, x^{n'} \ = \ \mu \sum_{n=2}^{\infty} a_{n-2}\, x^n . \qquad (11.44)
$$

Similarly by setting $n' = n + 1$, the sum $\sum_{n=0}^{\infty} a_n\, x^{n+1}$ becomes $\sum_{n'=1}^{\infty} a_{n'-1}\, x^{n'}$. Therefore,

$$
\nu \sum_{n=0}^{\infty} a_n\, x^{n+1} \ = \ \nu \sum_{n'=1}^{\infty} a_{n'-1}\, x^{n'} \ = \nu\, a_0\, x + \nu \sum_{n=2}^{\infty} a_{n-1}\, x^n \ , \qquad (11.45)
$$

and

$$
\rho \sum_{n=0}^{\infty} a_n\, x^n \ = \ \rho\, a_0 + \rho\, a_1\, x + \rho \sum_{n=2}^{\infty} a_n\, x^n . \qquad (11.46)
$$

By adding (11.44), (11.45), and (11.46), the last term on the right-hand side in (11.35), that is the current (11.43), becomes

$$
\begin{aligned}
&(\mu x^2 + \nu x + \rho) \sum_{n=0}^{\infty} a_n\, x^n \ = \ \mu \sum_{n=2}^{\infty} a_{n-2}\, x^n + \nu\, a_0\, x \\
&+ \nu \sum_{n=2}^{\infty} a_{n-1}\, x^n \ + \rho\, a_0 + \rho\, a_1\, x + \rho \sum_{n=2}^{\infty} a_n\, x^n \ . \qquad (11.47)
\end{aligned}
$$

In view of the foregoing differential equation (11.35)—which is the Frobenius version of equation(b)—is written as: [(11.36)] + [(11.42)] + [(11.47)] = 0. That is

$$\begin{aligned}
&\left[D^2 + (\alpha x^2 + \beta x + \gamma) D + (\mu x^2 + \nu x + \rho)\right] u(x) \\
&= \sum_{n=2}^{\infty} n(n-1)\, a_n\, x^{n-2} + (\alpha x^2 + \beta x + \gamma) \sum_{n=1}^{\infty} n\, a_n\, x^{n-1} \\
&\qquad + (\mu x^2 + \nu x + \rho) \sum_{n=0}^{\infty} a_n\, x^n \;=\; 0 \\
&= 2\, a_2 + 6\, a_3\, x + \sum_{n=2}^{\infty} (n+2)(n+1)\, a_{n+2}\, x^n + \alpha \sum_{n=2}^{\infty} (n-1)\, a_{n-1}\, x^n \\
&\quad + \beta\, a_1\, x + \beta \sum_{n=2}^{\infty} n\, a_n\, x^n + \gamma\, a_1 + 2\,\gamma\, a_2\, x + \gamma \sum_{n=2}^{\infty} (n+1)\, a_{n+1}\, x^n \\
&\quad + \mu \sum_{n=2}^{\infty} a_{n-2}\, x^n \;+ \nu \sum_{n=2}^{\infty} a_{n-1}\, x^n + \nu\, a_0\, x + \rho\, a_0 + \rho\, a_1\, x + \rho \sum_{n=2}^{\infty} a_n\, x^n \,.
\end{aligned}$$

Or more conveniently as (11.48) given below.

$$\begin{aligned}
&\left[D^2 + (\alpha x^2 + \beta x + \gamma) D + (\mu x^2 + \nu x + \rho)\right] u(x) \;=\; 0 \\
&= (\rho\, a_0 + 2\, a_2 + \gamma\, a_1) + x(\beta\, a_1 + \rho\, a_1 + 6\, a_3 + 2\gamma\, a_2 + \nu\, a_0) \\
&+ \sum_{n=2}^{\infty} x^n \{(n+2)(n+1)\, a_{n+2}\} \\
&+ \sum_{n=2}^{\infty} x^n \{\gamma(n+1) a_{n+1} + (\beta n + \rho)\, a_n + [\alpha(n-1) + \nu]\, a_{n-1} + \mu\, a_{n-2}\} \,. \qquad (11.48)
\end{aligned}$$

Equation (11.48) should hold for all values of x. And that can happen only if the coefficient multiplying any given power of x is zero: meaning, only if the following three equations, numbered (11.49), (11.50), (11.51), hold true. [Note: (11.49) refers to x^0, (11.50) to x^1, and (11.51) to x^n for all $n \geq 2$.]

$$(\rho\, a_0 + 2\, a_2 + \gamma\, a_1) \;=\; 0 \;, \qquad (11.49)$$

$$(\beta\, a_1 + \rho\, a_1 + 6\, a_3 + 2\gamma\, a_2 + \nu\, a_0) \;=\; 0 \;, \qquad (11.50)$$

$$\begin{aligned}
(n+2)(n+1)\, a_{n+2} \;=\; &-\gamma(n+1) a_{n+1} - (\beta n + \rho)\, a_n \\
&- [\alpha\,(n-1) + \nu]\, a_{n-1} - \mu\, a_{n-2} \,, \quad \textit{for all } n \geq 2 \,. \qquad (11.51)
\end{aligned}$$

Before proceeding further, it is important to recognize that the differential equation being solved here is second order and should have two independent constants. Unfortunately, (11.49) and (11.50) have four such constants: a_0, a_1, a_2, a_3. If a_2 and a_3 are eliminated, only two—that is, a_0 and a_1—will survive. And then the rest of a_n can be written in terms of these two. To that end, proceed as follows.

According to (11.49) and (11.50), we have

$$a_2 = -\left[\frac{\gamma a_1 + \rho a_0}{2}\right] \tag{11.52}$$

and

$$-\frac{(\beta+\rho)a_1 + \nu a_0 + 2\gamma a_2}{6} = -\frac{(\beta+\rho)a_1 + \nu a_0 - \gamma(\rho a_0 + \gamma a_1)}{6} = a_3 .$$

Therefore, (11.50) gives

$$a_3 = -\left[\frac{(\beta+\rho-\gamma^2)a_1 + (\nu - \gamma\rho)a_0}{6}\right] \quad . \tag{11.53}$$

Now that the variables a_2 and a_3 are available in terms of a_0 and a_1, the higher-order terms—namely $a_4, a_5, \ldots,$ etc.—may also be represented as functions of a_2 and a_3. To that purpose, setting $n = 2$ in (11.51) will lead to a_4, and $n = 3$ will yield a_5, and so on.

$$a_4 = -\left[\frac{\mu a_0 + (\alpha+\nu)a_1 + (2\beta+\rho)a_2 + 3\gamma a_3}{12}\right] . \tag{11.54}$$

Next use $n = 3$ in (11.51).

$$a_5 = -\left[\frac{\mu a_1 + (2\alpha+\nu)a_2 + (3\beta+\rho)a_3 + 4\gamma a_4}{20}\right] . \tag{11.55}$$

With $n = 4$ (11.51) gives

$$a_6 = -\left[\frac{\mu a_2 + (3\alpha+\nu)a_3 + (4\beta+\rho)a_4 + 5\gamma a_5}{30}\right] . \tag{11.56}$$

For $n = 5$, (11.51) leads to

$$a_7 = -\left[\frac{\mu a_3 + (4\alpha+\nu)a_4 + (5\beta+\rho)a_5 + 6\gamma a_6}{42}\right] . \tag{11.57}$$

Finally, the initial conditions must be satisfied. That is

$$u(x=0) \;=\; \sigma_1 \;=\; a_0 \quad ; \quad \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)_{x=0} \;=\; \sigma_2 \;=\; a_1 . \tag{11.58}$$

Combining (11.52) to (11.58) with (11.33) provide first few terms for the simple form of the Frobenius series solution for (11.32).

11.1.12 Examples Group II

Use the information given in (11.52)–(11.58) above and solve the following five differential equations similar in form to equation (b) (11.32). For simplicity, only the indices α - ρ are given below. [See (11.59)]

$$\begin{aligned}
II-(F) \;&: [\alpha=1\,;\;\beta=1\,;\;\gamma=1\,;\;\mu=1\,;\;\nu=1\,;\;\rho=1\,]\;.\\
II-(G) \;&: [\alpha=0\,;\;\beta=1\,;\;\gamma=2\,;\;\mu=3\,;\;\nu=4\,;\;\rho=5\,]\;.\\
II-(H) \;&: [\alpha=1\,;\;\beta=2\,;\;\gamma=3\,;\;\mu=4\,;\;\nu=5\,;\;\rho=6\,]\;.\\
II-(I) \;&: [\alpha=2\,;\;\beta=1\,;\;\gamma=2\,;\;\mu=3\,;\;\nu=2\,;\;\rho=2\,]\;.\\
II-(J) \;&: [\alpha=1\,;\;\beta=3\,;\;\gamma=3\,;\;\mu=1\,;\;\nu=1\,;\;\rho=2\,]\;.
\end{aligned} \tag{11.59}$$

Addition of (11.52)–(11.58) readily yields solutions of (11.59)-(F)–(J)— in terms of $a_0 \equiv \sigma_1$ and $a_1 \equiv \sigma_2$. Results are given below.

For II-(F), $\alpha=1\,;\beta=1\,;\gamma=1\,;\;\mu=1\,;\nu=1\,;\rho=1$. Therefore,

$$\begin{aligned}
II-(F)\;:\;u(x) \;=\; \sigma_1&\left[1-\frac{x^2}{2}+\frac{x^4}{24}+\frac{x^5}{15}-\frac{x^6}{720}-\frac{x^7}{70}\right]\\
+\;\sigma_2&\left[x-\frac{x^2}{2}-\frac{x^3}{6}+\left(\frac{7}{120}\right)x^5+\left(\frac{21}{720}\right)x^6-\left(\frac{43}{5040}\right)x^7\right]+O(x^8)\;.
\end{aligned}$$

For II-(G), $\alpha=0\,;\beta=1\,;\gamma=2\,;\;\mu=3\,;\nu=4\,;\rho=5$. Therefore,

$$\begin{aligned}
II-(G)\;:\;u(x) \;=\; \sigma_1&\left[1-\left(\frac{5}{2}\right)x^2+x^3+\left(\frac{17}{24}\right)x^4-\left(\frac{11}{60}\right)x^5\right]\\
+\sigma_1&\left[-\left(\frac{5}{144}\right)x^6-\left(\frac{43}{504}\right)x^7\right]+\sigma_2\left[x-x^2-\left(\frac{1}{3}\right)x^3+\left(\frac{5}{12}\right)x^4\right]\\
&+\sigma_2\left[\frac{x^5}{60}+\frac{x^6}{72}-\frac{x^7}{42}\right]+O(x^8)\;.
\end{aligned}$$

For II-(H), $\alpha = 1\,;\, \beta = 2\,;\, \gamma = 3\,;\, \mu = 4\,;\, \nu = 5\,;\, \rho = 6$. Therefore,

$$\begin{aligned} II-(H)\;:\; u(x) \;=\; & \sigma_1\left[1-3x^2+\left(\frac{13}{6}\right)x^3+\left(\frac{13}{24}\right)x^4-\left(\frac{23}{40}\right)x^5\right] \\ & +\sigma_1\left[-\left(\frac{103}{720}\right)x^6-\left(\frac{212}{5040}\right)x^7\right]+\sigma_2\left[x-\left(\frac{3}{2}\right)x^2+\left(\frac{1}{6}\right)x^3\right] \\ & +\sigma_2\left[\left(\frac{5}{8}\right)x^4-\left(\frac{3}{20}\right)x^5-\left(\frac{44}{720}\right)x^6-\left(\frac{335}{5040}\right)x^7\right]+O(x^8)\;. \end{aligned}$$

For II-(I), $\alpha = 2\,;\, \beta = 1\,;\, \gamma = 2\,;\, \mu = 3\,;\, \nu = 2\,;\, \rho = 2$. Therefore,

$$\begin{aligned} II-(I)\;:\; u(x) \;=\; & \sigma_1\left[1-x^2+\left(\frac{1}{3}\right)x^3-\frac{x^4}{12}+\frac{x^5}{4}\right] \\ & +\sigma_1\left[-\frac{x^6}{18}-\left(\frac{30}{1008}\right)x^7\right]+\sigma_2\left[x-x^2+\left(\frac{1}{6}\right)x^3-\left(\frac{x^4}{12}\right)\right] \\ & +\sigma_2\left[\left(\frac{17}{120}\right)x^5+\frac{x^6}{40}-\left(\frac{23}{1008}\right)x^7\right]+O(x^8)\;. \end{aligned}$$

For II-(J), $\alpha = 1\,;\, \beta = 3\,;\, \gamma = 3\,;\, \mu = 1\,;\, \nu = 1\,;\, \rho = 2$. Therefore,

$$\begin{aligned} II-(J)\;:\; u(x) \;=\; & \sigma_1\left[1-x^2+\left(\frac{5}{6}\right)x^3-\frac{x^4}{24}-\left(\frac{17}{60}\right)x^5\right] \\ & +\sigma_1\left[\frac{x^6}{12}+\left(\frac{323}{5040}\right)x^7\right]+\sigma_2\left[x-\left(\frac{3}{2}\right)x^2+\left(\frac{2}{3}\right)x^3+\frac{x^4}{3}\right] \\ & +\sigma_2\left[-\left(\frac{47}{120}\right)x^5+\frac{x^6}{720}+\left(\frac{516}{5040}\right)x^7\right]+O(x^8). \end{aligned}$$

11.1.13 Problems Group II

Use the information given in (11.52)–(11.58) and solve the following five differential equations similar in form to equation (b) (11.32). For simplicity, only the indices α to γ are given in (11.60) below.

$$\begin{aligned} & II.1\;:\; [\alpha = 1\,;\; \beta = 2\,;\; \gamma = 1\,;\; \mu = 3\,;\; \nu = 3\,;\; \rho = 1\,]\;. \\ & II.2\;:\; [\alpha = 0\,;\; \beta = 1\,;\; \gamma = 0\,;\; \mu = 2\,;\; \nu = 2\,;\; \rho = 0\,]\;. \\ & II.3\;:\; [\alpha = 1\,;\; \beta = 0\,;\; \gamma = 1\,;\; \mu = 2\,;\; \nu = 1\,;\; \rho = 1\,]\;. \\ & II.4\;:\; [\alpha = 0\,;\; \beta = 2\,;\; \gamma = 2\,;\; \mu = 0\,;\; \nu = 0\,;\; \rho = 1\,]\;. \\ & II.5\;:\; [\alpha = 2\,;\; \beta = 1\,;\; \gamma = 1\,;\; \mu = 5\,;\; \nu = 1\,;\; \rho = 4\,]\;. \end{aligned} \tag{11.60}$$

11.2 Frobenious Solution Around Regular Singular Point

According to an established *idea*[6.] and a recent *proof*[30.], if the differential equation

$$y''(x) + M(x)\,y'(x) + N(x)\,y(x) \;=\; 0 \tag{11.61}$$

has a 'Regular Singular Point,' say at $x = x_0$, then it has at least one solution that is expressible as a modified Taylor series of the form

$$y(x) \;=\; \sum_{n=0}^{\infty} a_n\,(x - x_0)^{(n+\nu_0)} \quad . \tag{11.62}$$

The unknown constant ν_0 is determined by substituting such series as solution to the given differential equation.

11.2.1 Equations of Type (c)

Equations similar to (11.63) that is given below will henceforth be referred to as the equations of Type (c). Here, as in Type (a) and Type (b) equations, all the given constants—namely β, γ, μ, ν, ρ—are real.

$$(c)\;:\; y''(x) + \left(\frac{\beta x + \gamma}{x}\right) y'(x) + \left(\frac{\mu x^2 + \nu x + \rho}{x^2}\right) y(x) \;=\; 0\,. \tag{11.63}$$

11.2.2 Solution

Clearly while $M(x)$,

$$M(x) \;=\; \left(\frac{\beta x + \gamma}{x}\right), \tag{11.64}$$

and $N(x)$,

$$N(x) \;=\; \left(\frac{\mu x^2 + \nu x + \rho}{x^2}\right), \tag{11.65}$$

are not analytic at $x = 0$, the products, $x\,M(x)$ and $x^2\,N(x)$,

$$\begin{aligned} x\,M(x) &= (\beta\,x + \gamma)\ , \\ x^2\,N(x) &= \left(\mu\,x^2 + \nu\,x + \rho\right)\ , \end{aligned} \tag{11.66}$$

are analytic at $x = 0$. As such $x = 0$ is a regular singular point. In accord with the Frobenius statement, the differential equation (11.63) is assured to have at least one solution that can be represented in the form of Maclaurin series (11.67) given below.

$$y(x) = \sum_{n=0}^{\infty} a_n\,x^{(n+\nu_0)}\ . \tag{11.67}$$

In order to solve the second-order differential equation (c) (11.63), the following two differentials of equation (11.67) are needed.

$$\begin{aligned} y'(x) &= \sum_{n=0}^{\infty} a_n\,(n+\nu_0)\,x^{(n+\nu_0-1)}\ , \\ y''(x) &= \sum_{n=0}^{\infty} a_n\,(n+\nu_0)(n+\nu_0-1)\,x^{(n+\nu_0-2)}\ . \end{aligned} \tag{11.68}$$

Combining (11.63), (11.67), and (11.68) gives

$$\begin{aligned} &\sum_{n=0}^{\infty} a_n\,(n+\nu_0)(n+\nu_0-1)\,x^{(n+\nu_0-2)} \\ &+\left(\frac{\beta\,x+\gamma}{x}\right)\sum_{n=0}^{\infty} a_n\,(n+\nu_0)\,x^{(n+\nu_0-1)} \\ &+\left(\frac{\mu\,x^2+\nu\,x+\rho}{x^2}\right)\sum_{n=0}^{\infty} a_n\,x^{(n+\nu_0)} = 0\ . \end{aligned} \tag{11.69}$$

Equations (11.63) and (11.69) can be put together in a more useful format as (11.70). This fact is owed to the Frobenius statement that a solution of (11.63) exists as a Maclaurin series of the form (11.67).

$$\begin{aligned} &\sum_{n=0}^{\infty} a_n\,\left[(n+\nu_0)(n+\nu_0+\gamma-1)+\rho\right]x^{(n+\nu_0-2)} \\ &+\sum_{n=0}^{\infty} a_n\,\left[\beta\,(n+\nu_0)+\nu\right]x^{(n+\nu_0-1)} \\ &+\sum_{n=0}^{\infty}[\mu]\,a_n\,x^{(n+\nu_0)} = 0\ . \end{aligned} \tag{11.70}$$

The first term of (11.70) can be written as

$$\begin{aligned}
&\sum_{n=0}^{\infty} a_n \left\{(n+\nu_0)(n+\nu_0+\gamma-1)+\rho\right\} x^{(n+\nu_0-2)} \\
&= a_0\, x^{(\nu_0-2)} \left\{\nu_0^2+\nu_0(\gamma-1)+\rho\right\} \\
&+ a_1\, x^{(\nu_0-1)} \left\{\nu_0^2+\nu_0(\gamma+1)+\gamma+\rho\right\} \\
&+ \sum_{n=2}^{\infty} a_n \left\{(n+\nu_0)(n+\nu_0+\gamma-1)+\rho\right\} x^{(n+\nu_0-2)} \quad . \qquad (11.71)
\end{aligned}$$

The second term of (11.70) can be transformed as follows. Set $n = n' - 1$ and then write the second term of (11.70) as

$$\begin{aligned}
&\sum_{n=0}^{\infty} a_n \left[\beta\,(n+\nu_0)+\nu\right] x^{(n+\nu_0-1)} \\
&= \sum_{n'=1}^{\infty} a_{n'-1} \left[\beta\,(n'-1+\nu_0)+\nu\right] x^{(n'+\nu_0-2)} \, . \qquad (11.72)
\end{aligned}$$

And finally change the variable n' back to the variable n.

$$\begin{aligned}
&\sum_{n=0}^{\infty} a_n \left[\beta\,(n+\nu_0)+\nu\right] x^{(n+\nu_0-1)} \\
&= \sum_{n=1}^{\infty} a_{n-1} \left[\beta\,(n-1+\nu_0)+\nu\right] x^{(n+\nu_0-2)} \\
&= a_0\,(\beta\,\nu_0+\nu)\, x^{(\nu_0-1)} + \sum_{n=2}^{\infty} a_{n-1} \left[\beta\,(n-1+\nu_0)+\nu\right] x^{(n+\nu_0-2)} \, . \qquad (11.73)
\end{aligned}$$

The third term of (11.70) can also be suitably transformed. To that end, set $n' = n + 2$ and proceed as follows.

$$\begin{aligned}
\sum_{n=0}^{\infty} [\mu]\, a_n\, x^{(n+\nu_0)} &= \sum_{n'=2}^{\infty} [\mu]\, a_{n'-2}\, x^{(n'+\nu_0-2)} \\
&= \sum_{n=2}^{\infty} [\mu]\, a_{n-2}\, x^{(n+\nu_0-2)} \, . \qquad (11.74)
\end{aligned}$$

By adding these terms, the original differential equation (11.63) is represented more neatly as (11.75) given below.

$$
\begin{aligned}
& a_0\, x^{(\nu_0-2)} \left\{\nu_0^2 + \nu_0(\gamma-1) + \rho\right\} \\
+\; & x^{(\nu_0-1)} \left\{a_1 \left[\nu_0^2 + \nu_0(\gamma+1) + \gamma + \rho\right] + a_0\,(\beta\,\nu_0 + \nu)\right\} \\
+\; & \sum_{n=2}^{\infty} a_n \left\{(n+\nu_0)(n+\nu_0+\gamma-1) + \rho\right\} x^{(n+\nu_0-2)} \\
+\; & \sum_{n=2}^{\infty} a_{n-1} \left[\beta\,(n-1+\nu_0) + \nu\right] x^{(n+\nu_0-2)} \\
+\; & \sum_{n=2}^{\infty} a_{n-2} \left[\mu\right] x^{(n+\nu_0-2)} \;=\; 0 \;. \qquad (11.75)
\end{aligned}
$$

If (11.75) is to hold for all x, coefficient of every power of x must be vanishing.

11.3 Indicial Equation

Indicial Equation is the Term with the Lowest Power of x in (11.75)

In order to properly satisfy the requirement that (11.75) holds true—namely that coefficients of every power of x in (11.75) is vanishing—we begin with the first term of (11.75) which has the lowest power of x—that is, the term $a_0\, x^{(\nu_0-2)}$—and set its coefficient equal to zero.

$$\left\{\nu_0^2 + \nu_0(\gamma-1) + \rho\right\} \;=\; 0 \;. \qquad (11.76)$$

Equation (11.76), being the term that multiplies the lowest power of x in (11.75), is labeled the 'Indicial Equation.'

11.4 Indicial Equation Roots

11.4.1 ν_1 and ν_2

The indicial equation always refers to a specified differential equation. In this case, the indicial equation (11.76) is particularized to the γ and ρ that appear in differential equation (11.63), or equivalently (11.75). And being a quadratic, it has two roots, ν_1 and ν_2.

Indicial Equation Roots Signify Possible Solutions of the Relevant Differential Equation

$$\begin{aligned} \nu_0 \;=\; \nu_1 \;&=\; -\left(\frac{\gamma-1}{2}\right)+\sqrt{\left(\frac{\gamma-1}{2}\right)^2-\rho}\;\;, \\ \nu_0 \;=\; \nu_2 \;&=\; -\left(\frac{\gamma-1}{2}\right)-\sqrt{\left(\frac{\gamma-1}{2}\right)^2-\rho}\;\;. \end{aligned} \tag{11.77}$$

The two roots of the indicial equation recorded in (11.77), namely $\nu_0 = \nu_1$ and $\nu_0 = \nu_2$, signify the two possible solutions of the original differential equation (11.63).

Category (1): Roots Differing by Non-integer

If the two roots, ν_1 and ν_2, of the indicial equation are unequal and their difference is not an integer, then the relevant second-order differential equation always posseses two linearly independent solutions, $y_1(x)$ and $y_2(x)$.

In this regard, consider the differential equation (11.78) given below.

$$y''(x)+\left(\frac{\beta x+\gamma}{x-x_0}\right)y'(x)+\left(\frac{\mu x^2+\nu x+\rho}{(x-x_0)^2}\right)y(x) \;=\; 0\;\;. \tag{11.78}$$

It has a regular singular point at $x = x_0$. And should its indicial equation have two roots that are unequal and differ by an integer. Meaning, if ν_1 and ν_2 are the two roots that are unequal and

$$|\,\nu_1-\nu_2\,| \;=\; 2\,|\sqrt{\left(\frac{\gamma-1}{2}\right)^2-\rho}\;|\quad, \tag{11.79}$$

is not an integer, then it will always posses two linearly independent solutions, $y_1(x)$ and $y_2(x)$, in the form of modified Taylor series:

$$y_1(x) = \sum_{n=0}^{\infty} a_n\,(x-x_0)^{(n+\nu_1)}\;\;. \tag{11.80}$$

$$y_2(x) = \sum_{n=0}^{\infty} a_n\,(x-x_0)^{(n+\nu_2)}\;\;. \tag{11.81}$$

Category (2): Roots Equal
First Solution Readily Found

If the two roots—ν_1 and ν_2—of the indicial equation are equal—meaning if $\nu_1 = \nu_2 \equiv \nu_0$ or, equivalently, $\rho = \left(\frac{\gamma-1}{2}\right)^2$—and the relevant differential equation—meaning (11.78)—has a regular singular point at $x = x_0$, then it will always posses a relatively easily accessible first solution, $y_1(x)$, in the form of modified Taylor series.

$$y_1(x) = \sum_{n=0}^{\infty} a_n \, (x - x_0)^{(n+\nu_0)} \quad . \tag{11.82}$$

Second Solution May Require More Effort

The second solution, $y_2(x)$, in contrast, may require more effort. This issue will be discussed later in greater detail.

Category (3): Roots Differing by an Integer

If the two roots, ν_1 and ν_2, of the indicial equation are unequal—say, with $\nu_1 > \nu_2$—and differ by an integer—meaning $\nu_1 - \nu_2$ is integral—and the relevant differential equation (11.78) has a regular singular point at $x = x_0$, then there always is a solution, say $y_1(x)$, that is of the form of a modified Taylor series.

$$y_1(x) = \sum_{n=0}^{\infty} a_n \, (x - x_0)^{(n+\nu_1)} \quad . \tag{11.83}$$

The second solution, say $y_2(x)$ that uses ν_2 instead of ν_1, may sometime diverge because of the presence of denominators that go to zero. Often, however, the arbitrary constant multiplying the whole solution may be changed to obtain non-divergent second solution. The details of such procedure will become clear when relevant solutions are discussed in what follows.

Term with the Second Lowest Power of x in (11.75)

Having set the term with the lowest power of x in (11.75) equal to zero, we consider next the second term—which carries the second lowest power of x, i.e., $x^{(\nu_0-1)}$—and set it equal to zero.

$$a_1 \left[\nu_0^2 + \nu_0(\gamma + 1) + \gamma + \rho\right] + a_0 \left(\beta \nu_0 + \nu\right) \; = \; 0 \quad . \tag{11.84}$$

Equation (11.84) gives

$$a_1 = -a_0 \left(\frac{\beta \nu_0 + \nu}{\left\{\nu_0^2 + \nu_0(\gamma - 1) + \rho\right\} + 2\,\nu_0 + \gamma} \right) = -a_0 \left(\frac{\beta \nu_0 + \nu}{2\,\nu_0 + \gamma} \right) \quad . \tag{11.85}$$

Remaining Three Terms of (11.75)

Finally, consider the remaining three terms of (11.75). They can be set together as a single sum of the form $\sum_{n=2}^{\infty} [...] x^{(n+\nu_0-2)}$.

$$\sum_{n=2}^{\infty} [\ \{a_n\,(n+\nu_0)(n+\nu_0+\gamma-1)+\rho\,\}$$
$$+\,a_{n-1}\,\{\beta\,(n-1+\nu_0)+\nu\,\}\ +a_{n-2}\,\{\mu\}]\,x^{(n+\nu_0-2)}\ =\ 0\ .$$

In order for this sum to add up identically to zero, terms multiplying each and every power of x must individually tend to zero. Therefore, the following equality

$$a_n\ =\ -\left[\frac{a_{n-1}\,[\beta\,(n-1+\nu_0)+\nu]+a_{n-2}\,[\mu]}{(n+\nu_0)(n+\nu_0+\gamma-1)+\rho}\right] \tag{11.86}$$

must obtain for all $n\ \geq\ 2$.
Points to note: Equation (11.85) gives a_1 in terms of a_0. Similarly, setting $n = 2, 3, 4, 5, 6, 7$ in (11.86) readily relates $a_7, a_6, a_5, a_4, a_3, a_2$ to a_0. Finally, unless otherwise stated, the parameter μ is being set equal to zero. This reduces the total number of parameters actively being used here to four.

11.4.2 Examples Group III

Indicial Equation Roots Unequal and Differing by Non-integer

Equations (11.88) describe a set of ten differential equations of the form

$$y''(x)+\left(\frac{\beta x+\gamma}{x}\right)y'(x)+\left(\frac{\nu x+\rho}{x^2}\right)y(x)\ =\ 0\,. \tag{11.87}$$

These equations are of type (c) where the parameter μ has been dropped. Note that these equations—(1) to (10)—have all a regular singular point at $x = 0$. Use (11.85). Set $n = 2, 3, 4, 5, 6, 7$ in (11.86). Thereby solve (11.88) (1) to (10). The relevant four parameters—namely β, γ, ν, ρ—are the following

$$III-(1)\ :\ [\beta=1\,;\ \gamma=1\,;\ \nu=4/3\,;\ \rho=-1/9\,]\ .$$
$$III-(2)\ :\ [\beta=0\,;\ \gamma=-1/2\,;\ \nu=-1/2\,;\ \rho=-5/2\,]\ .$$
$$III-(3)\ :\ [\beta=3\,;\ \gamma=-1\,;\ \nu=1\,;\ \rho=-5\,]\ .$$
$$III-(4)\ :\ [\beta=0\,;\ \gamma=-1/2\,;\ \nu=1/2\,;\ \rho=-5/2\,]\ .$$
$$III-(5)\ :\ [\beta=-1\,;\ \gamma=5/3\,;\ \nu=-2\,;\ \rho=-1/3\,]\ .$$
$$III-(6)\ :\ [\beta=1\,;\ \gamma=5/3\,;\ \nu=2\,;\ \rho=-1/3\,]\ .$$
$$III-(7)\ :\ [\beta=1\,;\ \gamma=-3/2\,;\ \nu=1/2\,;\ \rho=1\,]\ .$$

$$\begin{aligned} III-(8)\ &:\ [\beta=-1\,;\ \gamma=-3/2\,;\ \nu=-1/2\,;\ \rho=1\,]\ .\\ III-(9)\ &:\ [\beta=-1\,;\ \gamma=1\,;\ \nu=-4/3\,;\ \rho=-1/9\,]\ .\\ III-(10)\ &:\ [\beta=3\,;\ \gamma=-3\,;\ \nu=1\,;\ \rho=-4\,]\ . \end{aligned} \tag{11.88}$$

All the ten differential equations in (11.88) belong to category (1). Meaning that the two roots of their indicial equation are different, and the difference between the two roots is not an integer.

11.4.3 Solution

The solution to examples group III is worked out as follows.

Begin by treating equation III-(1) and keep track of the relevant parameters: that is $\beta=1;\ \gamma=1;\ \nu=4/3;\ \rho=-1/9$. Record the resultant differential equation.

$$y''(x)+\left(\frac{x+1}{x}\right)y'(x)+\left(\frac{\left(\frac{4}{3}\right)x-\left(\frac{1}{9}\right)}{x^2}\right)y(x)\ =\ 0\,.$$

Consider the indicial equation as described in (11.76).

$$\left\{\nu_0^2+\nu_0(\gamma-1)+\rho\right\}=\nu_0{}^2+\nu_0(1-1)-(1/9)=\ 0\ .$$

Note that the two roots of the indicial equation given above are $\nu_0\equiv\nu_1=(1/3)$ and $\nu_0\equiv\nu_2=(-1/3)$. They are different and their difference is not an integer. Work first with $\nu_0\equiv\nu_1=(1/3)$ and use (11.85) that relates a_1 to a_0.

$$a_1=-a_0\left(\frac{\beta\,\nu_0+\nu}{2\,\nu_0+\gamma}\right)\ =\ -\,a_0\left(\frac{\frac{1}{3}+\frac{4}{3}}{\frac{2}{3}+1}\right)\ =\ -\,a_0\ . \tag{11.89}$$

To calculate a_2 and higher-order a_n, for convenience, rewrite (11.86) in the form of (11.90) given below.

$$for\ \ n\geq 2\ ,\ \ a_n\ =\ -\,a_{n-1}\left[\frac{\beta\,(n-1+\nu_0)+\nu}{(n+\nu_0)(n+\nu_0+\gamma-1)+\rho}\right]\ . \tag{11.90}$$

For $n=2$, (11.90) gives

$$a_2\ =\ -\,a_1\left[\frac{(4/3)+(4/3)}{(7/3)(7/3)-(1/9)}\right]\ =\ -\,\frac{a_1}{2}\ =\ \frac{a_0}{2!}\ . \tag{11.91}$$

For $n = 3$, (11.90) leads to

$$a_3 = -a_2\left[\frac{(7/3)+(4/3)}{(10/3)(10/3)-(1/9)}\right] = -\frac{a_2}{3} = -\frac{a_0}{3!}\ . \tag{11.92}$$

For $n = 4$, (11.90) gives

$$a_4 = -a_3\left[\frac{(10/3)+(4/3)}{(13/3)(13/3)-(1/9)}\right] = -\frac{a_3}{4} = \frac{a_0}{4!}\ . \tag{11.93}$$

For $n = 5$, (11.90) gives

$$a_5 = -a_4\left[\frac{(13/3)+(4/3)}{(16/3)(16/3)-(1/9)}\right] = -\frac{a_4}{5} = -\frac{a_0}{5!}\ . \tag{11.94}$$

For $n = 6$, (11.90) leads to

$$a_6 = -a_5\left[\frac{(16/3)+(4/3)}{(19/3)(19/3)-(1/9)}\right] = -\frac{a_5}{6} = \frac{a_0}{6!}\ . \tag{11.95}$$

For $n = 7$, (11.90) gives

$$a_7 = -a_6\left[\frac{(19/3)+(4/3)}{(22/3)(22/3)-(1/9)}\right] = -\frac{a_6}{7} = -\frac{a_0}{7!}\ . \tag{11.96}$$

And so on, leading to the following result where the arbitrary constant a_0 is replaced either by arbitrary constant σ_1 or by σ_2.

$$\begin{aligned}(1a)\ : \left[\frac{y_1(x)}{\sigma_1 x^{\frac{1}{3}}}\right] &= 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \frac{x^6}{6!} - \frac{x^7}{7!} + \cdots \\ &= \exp(-x)\ .\end{aligned}$$

Similar procedure can now be followed for $\nu_0 \equiv \nu_2 = (-1/3)$, and we get

$$\begin{aligned}(1b)\ : \left[\frac{y_2(x)}{\sigma_2 x^{\frac{-1}{3}}}\right] &= 1 - 3x + \left(\frac{3^2}{4}\right)x^2 - \left(\frac{3^3}{4\cdot 7}\right)x^3 + \left(\frac{3^4}{4\cdot 7\cdot 10}\right)x^4 \\ &\quad - \left(\frac{3^5}{4\cdot 7\cdot 10\cdot 13}\right)x^5 + \left(\frac{3^6}{4\cdot 7\cdot 10\cdot 13\cdot 16}\right)x^6 \\ &\quad - \left(\frac{3^7}{4\cdot 7\cdot 10\cdot 13\cdot 16\cdot 19}\right)x^7 + \cdots\end{aligned}$$

For equation III-(2), $\beta = 0\,; \gamma = -1/2\,; \nu = -1/2\,; \rho = -5/2$ and the solutions are

$$III-(2a)\ :\left[\frac{y_1(x)}{\sigma_1 x^{\frac{5}{2}}}\right] = 1+\frac{x}{9}+\frac{x^2}{198}+\frac{x^3}{7,722}+\frac{x^4}{463,320}+\frac{x^5}{39,382,200} + \frac{x^6}{4,489,570,800}+\frac{x^7}{659,966,907,600}+O(x^8)$$

$$III-(2b)\ :\left[\frac{y_2(x)}{\sigma_2 x^{-1}}\right] = 1-\frac{x}{5}+\frac{x^2}{30}-\frac{x^3}{90}-\frac{x^4}{360}-\frac{x^5}{5400} - \frac{x^6}{162,000}-\frac{x^7}{7,938,000}+\cdots$$

For III-(3), $\beta = 3\,; \gamma = -1\,; \nu = 1\,; \rho = -5$. Therefore, the two solutions

$$III-(3a)\ :\left[\frac{y_1(x)}{\sigma_1 x^{1+\sqrt{6}}}\right] = 1-\frac{x}{23}(32+5\sqrt{6})+\frac{x^2}{460}(472+183\sqrt{6}) - \frac{x^3}{2070}(1491+629\sqrt{6})+\frac{x^4}{288}(126+47\sqrt{6})+\frac{x^5}{720}(-375+37\sqrt{6}) + \frac{x^6}{8640}(5615-1731\sqrt{6})+\frac{x^7}{1512000}(-901448+333403\sqrt{6})+\cdots$$

$$III-(3b)\ :\left[\frac{y_2(x)}{\sigma_2 x^{1-\sqrt{6}}}\right] = 1+\frac{x}{23}(-32+5\sqrt{6})+\frac{x^2}{460}(472-183\sqrt{6}) + \frac{x^3}{2070}(-1491+629\sqrt{6})+\frac{x^4}{288}(126-47\sqrt{6})-\frac{x^5}{720}(375+37\sqrt{6}) + \frac{x^6}{8640}(5615+1731\sqrt{6})-\frac{x^7}{1512000}(901448+333403\sqrt{6})+\cdots$$

For III-(4), $\beta = 0\,; \gamma = -1/2\,; \nu = 1/2\,; \rho = -5/2$ and the solutions are

$$III-(4a)\ :\left[\frac{y_1(x)}{\sigma_1 x^{\frac{5}{2}}}\right] = 1-\left(\frac{x}{9}\right)+\left(\frac{x^2}{9\cdot 22}\right)-\left(\frac{x^3}{9\cdot 22\cdot 39}\right)+\left(\frac{x^4}{9\cdot 22\cdot 39\cdot 60}\right) - \left(\frac{x^5}{9\cdot 22\cdot 39\cdot 60\cdot 85}\right)+\left(\frac{x^6}{9\cdot 22\cdot 39\cdot 60\cdot 85\cdot 114}\right) - \left(\frac{x^7}{9\cdot 22\cdot 39\cdot 60\cdot 85\cdot 114\cdot 147}\right)+\cdots$$

$$III-(4b)\ :\left[\frac{y_2(x)}{\sigma_2 x^{-1}}\right]\ =\ 1+\left(\frac{x}{5}\right)+\left(\frac{x^2}{5\cdot 6}\right)+\left(\frac{x^3}{5\cdot 6\cdot 3}\right)-\left(\frac{x^4}{5\cdot 6\cdot 3\cdot 4}\right)$$
$$+\qquad\left(\frac{x^5}{5\cdot 6\cdot 3\cdot 4\cdot 15}\right)-\left(\frac{x^6}{5\cdot 6\cdot 3\cdot 4\cdot 15\cdot 30}\right)+\left(\frac{x^7}{5\cdot 6\cdot 3\cdot 4\cdot 15\cdot 30\cdot 49}\right)+\cdots$$

For III-(5), $\beta=-1\,;\,\gamma=5/3\,;\,\nu=-2\,;\,\rho=-1/3$ and the solutions are

$$III-(5a):\left[\frac{y_1(x)}{\sigma_1 x^{\frac{1}{3}}}\right]=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\frac{x^6}{6!}+\frac{x^7}{7!}+\cdots$$
$$=\exp(x)\ .$$

$$III-(5b)\ :\left[\frac{y_2(x)}{\sigma_2 x^{-1}}\right]\ =\ 1-3x-\left(\frac{3^2}{2}\right)x^2-\left(\frac{3^3}{2\cdot 5}\right)x^3-\left(\frac{3^4}{2\cdot 5\cdot 8}\right)x^4$$
$$-\qquad\left(\frac{3^5}{2\cdot 5\cdot 8\cdot 11}\right)x^5-\left(\frac{3^6}{2\cdot 5\cdot 8\cdot 11\cdot 14}\right)x^6-\left(\frac{3^7}{2\cdot 5\cdot 8\cdot 11\cdot 14\cdot 17}\right)x^7+...$$

For III-(6), $\beta=1\,;\,\gamma=5/3\,;\,\nu=2\,;\,\rho=-1/3$ and the solutions are

$$III-(6a):\left[\frac{y_1(x)}{\sigma_1 x^{\frac{1}{3}}}\right]=1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}-\frac{x^5}{5!}+\frac{x^6}{6!}-\frac{x^7}{7!}+\cdots$$
$$=\exp(-x)\ .$$

$$III-(6b)\ :\left[\frac{y_2(x)}{\sigma_2 x^{-1}}\right]\ =\ 1+3x-\left(\frac{3^2}{2}\right)x^2+\left(\frac{3^3}{2\cdot 5}\right)x^3-\left(\frac{3^4}{2\cdot 5\cdot 8}\right)x^4$$
$$+\qquad\left(\frac{3^5}{2\cdot 5\cdot 8\cdot 11}\right)x^5-\left(\frac{3^6}{2\cdot 5\cdot 8\cdot 11\cdot 14}\right)x^6+\left(\frac{3^7}{2\cdot 5\cdot 8\cdot 11\cdot 14\cdot 17}\right)x^7+\cdots$$

For III-(7), $\beta=1\,;\,\gamma=-3/2\,;\,\nu=1/2\,;\,\rho=1$ and the solutions are

$$III-(7a)\ :\left[\frac{y_1(x)}{\sigma_1 x^2}\right]\ =\ 1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}-\frac{x^5}{5!}+\frac{x^6}{6!}-\frac{x^7}{7!}+\cdots$$
$$=\qquad\exp(-x)\ .$$

$$III-(7b)\ :\left[\frac{y_2(x)}{\sigma_2 x^{1/2}}\right]\ =\ 1+2x-\left(\frac{2^2}{1}\right)x^2+\left(\frac{2^3}{1\cdot 3}\right)x^3-\left(\frac{2^4}{1\cdot 3\cdot 5}\right)x^4$$
$$+\qquad\left(\frac{2^5}{1\cdot 3\cdot 5\cdot 7}\right)x^5-\left(\frac{2^6}{1\cdot 3\cdot 5\cdot 7\cdot 9}\right)x^6+\left(\frac{2^7}{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11}\right)x^7+\cdots$$

For III-(8), $\beta = -1\,;\, \gamma = -3/2\,;\, \nu = -1/2\,;\, \rho = 1$ and the solutions are

$$III-(8a):\left[\frac{y_1(x)}{\sigma_1 x^2}\right] = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots$$
$$= \exp(x)\ .$$

$$III-(8b):\left[\frac{y_2(x)}{\sigma_2 x^{1/2}}\right] = 1 - 2x - \left(\frac{2^2}{1}\right)x^2 - \left(\frac{2^3}{1\cdot 3}\right)x^3 - \left(\frac{2^4}{1\cdot 3\cdot 5}\right)x^4$$
$$- \left(\frac{2^5}{1\cdot 3\cdot 5\cdot 7}\right)x^5 - \left(\frac{2^6}{1\cdot 3\cdot 5\cdot 7\cdot 9}\right)x^6 - \left(\frac{2^7}{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11}\right)x^7 + \cdots$$

For III-(9), $\beta = -1\,;\, \gamma = 1\,;\, \nu = -4/3\,;\, \rho = -1/9$ and the solutions are

$$III-(9a):\left[\frac{y_1(x)}{\sigma_1 x^{\frac{1}{3}}}\right] = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots$$
$$= \exp(x)\ .$$

$$III-(9b):\left[\frac{y_2(x)}{\sigma_2 x^{-\frac{1}{3}}}\right] = 1 + \left(\frac{3^1}{1}\right)x + \left(\frac{3^2}{1\cdot 4}\right)x^2 + \left(\frac{3^3}{1\cdot 4\cdot 7}\right)x^3$$
$$+ \left(\frac{3^4}{1\cdot 4\cdot 7\cdot 10}\right)x^4 + \left(\frac{3^5}{1\cdot 4\cdot 7\cdot 10\cdot 13}\right)x^5 + \left(\frac{3^6}{1\cdot 4\cdot 7\cdot 10\cdot 13\cdot 16}\right)x^6$$
$$+ \left(\frac{3^7}{1\cdot 4\cdot 7\cdot 10\cdot 13\cdot 16\cdot 19}\right)x^7 + \cdots$$

For III-(10), $\beta = 3\,;\, \gamma = -3\,;\, \nu = 1\,;\, \rho = -4$ and the solutions are

$$III-(10a):\left[\frac{y_1(x)}{\sigma_1 x^{2+2\sqrt{2}}}\right] = 1 - \frac{x}{31}(41 + 22\sqrt{2})$$
$$+ \frac{x^2}{62}(85 + 63\sqrt{2}) - \frac{x^3}{4,278}(5,049 + 3,457\sqrt{2})$$
$$+ \frac{x^4}{1,488}(1,064 + 797\sqrt{2}) - \frac{x^5}{52,080}(23,316 + 11,485\sqrt{2})$$
$$+ \frac{x^6}{312,480}(191,026 - 61,923\sqrt{2}) + \frac{x^7}{312,480}(-264,230 + 159,191\sqrt{2}) + \cdots$$

$$III-(10b):\left[\frac{y_2(x)}{\sigma_2 x^{2-2\sqrt{2}}}\right] = 1 + \frac{x}{31}(-41 + 22\sqrt{2}) + \frac{x^2}{62}(85 - 63\sqrt{2})$$

$$+ \quad \frac{x^3}{4,278}(-5,049+3,457\sqrt{2})$$
$$+ \quad \frac{x^4}{1,488}(1,064-797\sqrt{2}) + \frac{x^5}{52,080}(-23,316+11,485\sqrt{2})$$
$$+ \quad \frac{x^6}{312,480}(191,026+61,923\sqrt{2}) - \frac{x^7}{312,480}(264,230+159,191\sqrt{2}) + \cdots$$

11.4.4 Problems Group III

Indicial Equation Roots Unequal and Differing by Non-integer

Use (11.85) and set $n = 2, 3, 4, 5, 6, 7$ in (11.86), and solve the following differential equations that are of the form (11.78) and belong to category (1). For simplicity, the index μ has been set equal to zero. The other four indices, β, γ, ν, ρ, are given in (11.97) (1)–(5) below.

$$\begin{aligned}
&(1) \ : \ [\beta = -3 \ ; \ \gamma = 3 \ ; \ \nu = 2 \ ; \ \rho = -4 \,] \ . \\
&(2) \ : \ [\beta = 2 \ ; \ \gamma = -3 \ ; \ \nu = 1 \ ; \ \rho = -2 \,] \ . \\
&(3) \ : \ [\beta = 3 \ ; \ \gamma = -2 \ ; \ \nu = -2 \ ; \ \rho = -3 \,] \ . \\
&(4) \ : \ [\beta = 1 \ ; \ \gamma = -2 \ ; \ \nu = -1 \ ; \ \rho = -1 \,] \ . \\
&(5) \ : \ [\beta = -1 \ ; \ \gamma = -5 \ ; \ \nu = -1 \ ; \ \rho = -4 \,] \ .
\end{aligned} \tag{11.97}$$

11.5 Examples Group IV

Indicial Equation Roots Are Equal

Listed in (11.99) are eight differential equations in the form of (11.98).

$$y''(x) + \left(\frac{\beta x + \gamma}{x}\right) y'(x) + \left(\frac{\nu x + \rho}{x^2}\right) y(x) \ = \ 0 \, . \tag{11.98}$$

Equations (A)–(H) are all of category (2): meaning, the two roots, ν_1 and ν_2, of their indicial equation,

$$\nu_0^2 + \nu_0(\gamma - 1) + \rho \ = \ 0 \ ,$$

are equal—that is $\nu_1 \to \nu_0$ and $\nu_2 \to \nu_0$. Note also that these differential equations have a regular singular point at $x = 0$. The parameter μ has been set equal to zero in all of these equations. The other four parameters—namely β, γ, ν, ρ—are listed in

a simple format in (11.99) below.

$$\begin{aligned}
&(A) \;:\; [\beta = -3\,;\; \gamma = 1\,;\; \nu = -3\,;\; \rho = 0\,]\;. \\
&(B) \;:\; [\beta = 4\,;\; \gamma = 1\,;\; \nu = -1\,;\; \rho = 0\,]\;. \\
&(C) \;:\; [\beta = -2\,;\; \gamma = 3\,;\; \nu = -4\,;\; \rho = 1\,]\;. \\
&(D) \;:\; [\beta = 1\,;\; \gamma = -3\,;\; \nu = 1\,;\; \rho = 4\,]\;. \\
&(E) \;:\; [\beta = -1\,;\; \gamma = -1\,;\; \nu = -1\,;\; \rho = 1\,]\;. \\
&(F) \;:\; [\beta = 1\,;\; \gamma = 2\,;\; \nu = 2\,;\; \rho = 1/4\,]\;. \\
&(G) \;:\; [\beta = 1\,;\; \gamma = -1\,;\; \nu = 1\,;\; \rho = 1\,]\;. \\
&(H) \;:\; [\beta = 1\,;\; \gamma = -2\,;\; \nu = 1\,;\; \rho = 9/4\,]\;.
\end{aligned} \tag{11.99}$$

11.5.1 Solution: (11.99)

As stated above, the two roots of the indicial equation, that is

$$\nu_0^2 + \nu_0(\gamma - 1) + \rho \;=\; 0\;,$$

relating to equations given in (11.99)-(A) to -(H), are equal. Label this double-root ν_0. Use (11.85). Then set $n = 2, 3, 4, 5, 6, 7$ in (11.86). And thereby determine the first solution for each of the eight differential equations listed in (11.99).

The method for working out the first solution of these differential equations is quite straightforward. Indeed, it is exactly the same method used earlier for the examples set III. The only difference here is that for (11.99)-(A), similar to Piaggio's *work*[10.] the first solution is being written entirely in terms of the variable ν_0. And when so written, the first solution will also be called the general solution which is described in detail in the following section. Numerical representation—which is done by replacing the variable ν_0 with its value equal to the relevant double root of the indicial equation—will be made at the end of the current process. Equation (11.85), that is similar to (11.100) below, relates a_1 to a_0.

$$a_1 \;=\; -\,a_0 \left(\frac{\beta\,\nu_0 + \nu}{2\,\nu_0 + \gamma} \right)\;, \tag{11.100}$$

For $n \geq 2$, the relevant equation is (11.86) that is similar to (11.101).

$$a_n = -a_{n-1} \left[\frac{\beta\,(n - 1 + \nu_0) + \nu}{(n + \nu_0)(n + \nu_0 + \gamma - 1) + \rho} \right]\;. \tag{11.101}$$

11.5.2 General Solution: (11.99)-A

Henceforth, in this book, the statement general solution of a given differential equation will imply a particular form of its first solution that is written as a function of the variable ν_0. The general solution is determined as follows.

First one considers the indicial equation and determines its roots. In relation to (11.99)-(*A*) , where $\beta = -3, \gamma = 1, \nu = -3, \rho = 0$, the indicial equation,

$$\nu_0^2 + \nu_0(\gamma - 1) + \rho \;=\; \nu_0^2 \;=\; 0 \;,$$

has two roots, $\nu_0 = \nu_1 = 0$ and $\nu_0 = \nu_2 = 0$. And both the roots are equal to zero.

Second, one identifies the relevant differential equation: it is (11.99)-(A)—written below as (11.102)—that is obtained from (11.98) by using the given values of the parameters: $\beta = -3, \gamma = 1, \nu = -3, \rho = 0$.

$$y''(x) + \left(\frac{1-3\,x}{x}\right) y'(x) - \left(\frac{3}{x}\right) y(x) \;=\; 0 \tag{11.102}$$

Third, one uses (11.100).

$$a_1 \;=\; -\,a_0 \left(\frac{\beta\,\nu_0 + \nu}{2\,\nu_0 + \gamma}\right) \;=\; 3\,a_0 \left(\frac{1+\nu_0}{1+2\,\nu_0}\right) . \tag{11.103}$$

Fourth, for $n \geq 2$, one writes (11.101) as

$$\begin{aligned} a_n &= -a_{n-1} \left[\frac{\beta\,(n-1+\nu_0) + \nu}{(n+\nu_0)(n+\nu_0+\gamma-1) + \rho}\right] \\ &= a_{n-1} \left[\frac{3}{(n+\nu_0)}\right] . \end{aligned} \tag{11.104}$$

As such

$$\begin{aligned} a_2 &= a_1 \left[\frac{3}{(2+\nu_0)}\right] \;=\; a_0 \left[\frac{3^2\,(1+\nu_0)}{(2+\nu_0)(1+2\nu_0)}\right] , \\ a_3 &= a_2 \left[\frac{3}{(3+\nu_0)}\right] \;=\; a_0 \left[\frac{3^3\,(1+\nu_0)}{(2+\nu_0)(3+\nu_0)(1+2\nu_0)}\right] , \\ a_4 &= a_3 \left[\frac{3}{(4+\nu_0)}\right] \;=\; a_0 \left[\frac{3^4\,(1+\nu_0)}{(2+\nu_0)(3+\nu_0)(4+\nu_0)(1+2\nu_0)}\right] , \\ a_5 &= a_4 \left[\frac{3}{(5+\nu_0)}\right] \;=\; a_0 \left[\frac{3^5\,(1+\nu_0)}{(2+\nu_0)(3+\nu_0)(4+\nu_0)(5+\nu_0)(1+2\nu_0)}\right] , \\ a_6 &= a_0 \left[\frac{3^6\,(1+\nu_0)}{(2+\nu_0)(3+\nu_0)(4+\nu_0)(5+\nu_0)(6+\nu_0)(1+2\nu_0)}\right] , \end{aligned}$$

$$a_7 = a_0 \left[\frac{3^7 (1+\nu_0)}{(2+\nu_0)(3+\nu_0)(4+\nu_0)(5+\nu_0)(6+\nu_0)(7+\nu_0)(1+2\nu_0)} \right]. \tag{11.105}$$

The first solution, $Y_{A;1}(x, \nu_0)$, is all contained in (11.103) and (11.105). As stated above, this solution—meaning (11.106)—will henceforth also be called the general solution, $G_A(x, \nu_0)$, of (11.99)-(*A*) or equivalently of (11.102).

$$\begin{aligned}
\frac{G_A(x,\nu_0)}{\sigma_1} &= Y_{A;1}(x,\nu_0) \\
&= x^{\nu_0}\,[1 + \frac{3\,(1+\nu_0)\,x}{(1+2\,\nu_0)} + \frac{3^2\,(1+\nu_0)\,x^2}{(2+\nu_0)(1+2\nu_0)} + \frac{3^3\,(1+\nu_0)\,x^3}{(2+\nu_0)(3+\nu_0)(1+2\nu_0)} \\
&+ \frac{3^4\,(1+\nu_0)\,x^4}{(2+\nu_0)(3+\nu_0)(4+\nu_0)(1+2\nu_0)} \\
&+ \frac{3^5\,(1+\nu_0)\,x^5}{(2+\nu_0)(3+\nu_0)(4+\nu_0)(5+\nu_0)(1+2\nu_0)} \\
&+ \frac{3^6\,(1+\nu_0)\,x^6}{(2+\nu_0)(3+\nu_0)(4+\nu_0)(5+\nu_0)(6+\nu_0)(1+2\nu_0)} \\
&+ \frac{3^7\,(1+\nu_0)\,x^7}{(2+\nu_0)(3+\nu_0)(4+\nu_0)(5+\nu_0)(6+\nu_0)(7+\nu_0)(1+2\nu_0)}] + O(x^8)
\end{aligned} \tag{11.106}$$

The double root for (11.99)-(*A*), i.e. (11.102), is $\nu_0 = 0$. Therefore, the first solution, $Y_{A;1}(x, \nu_0)$, for (11.99)-(*A*) is immediately available in numerical format $Y_{A;1}(x, \nu_0 = 0)$, by setting $\nu_0 = 0$ in its general solution (11.106). [See, for instance, (11.107).]

$$\begin{aligned}
(A): \left[\frac{Y_{A;1}(x,\nu_0=0)}{\sigma_1}\right] &\equiv \left[\frac{Y_{A;1}(x,0)}{\sigma_1}\right] = \frac{G_A(x,\nu_0=0)}{\sigma_1} \\
&= x^0[1 + 3\,x + \frac{9\,x^2}{2} + \frac{9\,x^3}{2} + \frac{27\,x^4}{8} + \frac{81\,x^5}{40} + \frac{81\,x^6}{80} + \frac{243\,x^7}{560} + \cdots] \\
&= x^0\,\exp(3x) = \exp(3x).
\end{aligned} \tag{11.107}$$

For $Y_{A;1}(x, 0)$, rather than σ_0, it was convenient to use σ_1 as the arbitrary constant. Similarly, for the second solution $Y_{A;2}(x, 0)$—to be studied later—σ_2 would be the relevant arbitrary constant.

11.6 First Solution of (11.99)-(B)→(E)

First solution of differential equations (*B*), (*C*), (*D*), (*E*), (*F*), (*G*), and (*H*) can be found in similar fashion and the results are as follows.

For differential equation (B), $\beta = 4$, $\gamma = 1$, $\nu = -1$ and $\rho = 0$. The double root of the indicial equation is again $\nu_0 = 0$ and the first solution is

$$(B) : \left[\frac{Y_{B;1}(x,0)}{\sigma_1 x^0}\right] = 1 + x - \frac{3\,x^2}{4} + \frac{7\,x^3}{12} - \frac{77\,x^4}{192} + \frac{77\,x^5}{320} - \frac{1,463\,x^6}{11,520} + \frac{4,807\,x^7}{80,640} + \cdots \tag{11.108}$$

For differential equation (C), $\beta = -2$, $\gamma = 3$, $\nu = -4$ and $\rho = 1$, the double root of the indicial equation is $\nu_0 = -1$ and its first solution is

$$(C) : \left[\frac{Y_{C;1}(x,-1)}{\sigma_1 x^{-1}}\right] = 1 + 2\,x + 2\,x^2 + \frac{4\,x^3}{3} + \frac{2\,x^4}{3} + \frac{4\,x^5}{15} + \frac{4\,x^6}{45} + \frac{8\,x^7}{315} + \cdots = \exp(2x)\ . \tag{11.109}$$

For differential equation (D), $\beta = 1$, $\gamma = -3$, $\nu = 1$ and $\rho = 4$, the double root of the indicial equation is $\nu_0 = 2$ and its first solution is

$$(D) : \left[\frac{Y_{D;1}(x,2)}{\sigma_1\, x^2}\right] = \quad 1 - 3x + 3\,x^2 - \frac{5\,x^3}{3} + \frac{5\,x^4}{8} - \frac{7\,x^5}{40} + \frac{7\,x^6}{180} - \frac{x^7}{140} + \cdots \tag{11.110}$$

For differential equation (E), $\beta = -1$, $\gamma = -1$, $\nu = -1$ and $\rho = 1$, the double root of the indicial equation is $\nu_0 = 1$ and its first solution is

$$(E) : \left[\frac{Y_{E;1}(x,1)}{\sigma_1\, x}\right] = \quad 1 + 2x + \frac{3\,x^2}{2} + \frac{2\,x^3}{3} + \frac{5\,x^4}{24} + \frac{x^5}{20} + \frac{7\,x^6}{720} + \frac{x^7}{630} + \cdots \tag{11.111}$$

For differential equation (F), $\beta = 1$, $\gamma = 2$, $\nu = 2$ and $\rho = 1/4$, the double root of the indicial equation is $\nu_0 = -1/2$ and its first solution is

$$(F) : \left[\frac{Y_{F;1}(x,-1/2)}{\sigma_1\, x^{-\frac{1}{2}}}\right] = 1 - \frac{3x}{2} + \frac{15\,x^2}{16} - \frac{35\,x^3}{96} + \frac{105\,x^4}{1,024} - \frac{231x^5}{10,240} + \frac{1,001\,x^6}{245,760} - \frac{143x^7}{229,376} + \cdots \tag{11.112}$$

For differential equation (G), $\beta = 1$, $\gamma = -1$, $\nu = 1$ and $\rho = 1$, the double root of the indicial equation is $\nu_0 = 1$. Later on, both its first and second solutions will be worked out and details presented—as, for instance, in (11.123) and (11.132). In the meantime the first solution is given below.

$$(G): \left[\frac{Y_{G;1}(x,1)}{\sigma_1 x}\right] = 1 - 2x + \frac{3x^2}{2} - \frac{2x^3}{3} + \frac{5x^4}{24} - \frac{x^5}{20} + \frac{7x^6}{720} - \frac{x^7}{630} + \cdots$$
$$= (1-x)\exp(-x) \tag{11.113}$$

Finally, for differential equation (H), $\beta = 1, \gamma = -2, \nu = 1$ and $\rho = 9/4$, the double root of the indicial equation is $\nu_0 = 3/2$ and its first solution is

$$(H): \left[\frac{Y_{H;1}(x,3/2)}{\sigma_1 x^{\frac{3}{2}}}\right] = 1 - \frac{5x}{2} + \frac{35x^2}{16} - \frac{35x^3}{32} + \frac{385x^4}{1,024} - \frac{1,001x^5}{10,240}$$
$$+ \frac{1,001x^6}{49,152} - \frac{2,431x^7}{688,128} + \cdots. \tag{11.114}$$

11.7 Methodology For Second Solution

Equation (11.99)-(A)–(H) are of category (2): Meaning their indicial equations have double roots. The methodology to be used for working out their second solution will involve two different methods: one inspired by a method used by *Piaggio*[10.] and the other based on what is known as the 'method of order reduction.' The relevant details are provided in the following. For convenience, these methods are termed as the 'second methods.'

11.7.1 Piaggio-Like Solution

Piaggio, while studying a particular differential equation whose indicial equation roots are equal, came up with an idea about the second solution. Assuming that his idea holds true for the differential equation being studied here, the two solutions of a category (2) differential equation should have the following characteristics.

The first solution is the general solution itself while the second solution is equal to the differential with respect to ν_0 of the general solution. At the end of the process, one needs to convert the first- and the second solution into numerical format. This is done by replacing the symbol ν_0 by its actual value: which is the relevant double root of the indicial equation. This procedure will become clear as the process unfolds.

Consider differential equation (11.99)-(A), its general solution—see (11.106)—$G_A(x, \nu_0)$, and proceed as follows.

$$Y_{A;1}(x, \nu_0) \;=\; G_A(x, \nu_0), \tag{11.115}$$

and

$$Y_{A;2}(x,\nu_0) \;=\; \left[\frac{d\,Y_{A;1}(x,\nu_0)}{d\,\nu_0}\right] \;=\; \left[\frac{d\,G_A(x,\nu_0)}{d\,\nu_0}\right]. \tag{11.116}$$

Because $Y_{A;1}(x,\nu_0)$ includes the term x^{ν_0}, the differential of x^{ν_0} needs to be worked out. To that end, define a symbol $X(\nu_0,x)$.

$$X(\nu_0,x) = x^{\nu_0} \;. \tag{11.117}$$

And proceed as follows. Take logarithm of both sides.

$$\log X(\nu_0,x) = \log(x^{\nu_0}) \;=\; \nu_0\,\log(x) \;, \tag{11.118}$$

and differentiate with respect to ν_0.

$$\frac{1}{X(\nu_0,x)}\cdot\left(\frac{\mathrm{d}x(\nu_0,x)}{d\nu_0}\right) = \log(x) \;, \tag{11.119}$$

Note: Any remaining terms that would arise if x itself depended on ν_0 are not included in (11.119). Now rewrite the above as

$$\frac{\mathrm{d}x(\nu_0,x)}{d\nu_0} = X(\nu_0,x)\;\log(x) \;. \tag{11.120}$$

and replace $X(\nu_0,x)$ by x^{ν_0}.

$$\left(\frac{\mathrm{d}x^{\nu_0}}{d\nu_0}\right) = x^{\nu_0}\;\log(x) \;. \tag{11.121}$$

Accordingly, the second solution is

$$\begin{aligned}
&Y_{A;2}(x,\nu_0) \;\equiv\; \left[\frac{d\,Y_{A;1}(x,\nu_0)}{d\,\nu_0}\right] \;=\; \left[\frac{d\,G_A(x,\nu_0)}{d\,\nu_0}\right] \\
&= Y_{A;1}(x,\nu_0)\cdot\log(x) - \sigma_1\,x^{\nu_0}\left[\frac{3\,x}{(1+2\,\nu_0)^2}+\frac{3\,x^2}{(2+\nu_0)^2}+\frac{6\,x^2}{(1+2\nu_0)^2}\right] \\
&- \sigma_1\,x^{\nu_0}\left[27\frac{\left(11+22\nu_0+17{\nu_0}^2+4{\nu_0}^3\right)}{(2+\nu_0)^2(3+\nu_0)^2(1+2\nu_0)^2}\right]x^3 \\
&- \sigma_1 x^{\nu_0}\left[162\frac{(25+61\nu_0+59{\nu_0}^2+23{\nu_0}^3+3{\nu_0}^4)}{(2+\nu_0)^2(3+\nu_0)^2(4+\nu_0)^2(1+2\nu_0)^2}\right]x^4 \\
&- \sigma_1 x^{\nu_0}\left[243\frac{(274+758\nu_0+847{\nu_0}^2+428{\nu_0}^3+97{\nu_0}^4)+8{\nu_0}^5)}{(2+\nu_0)^2(3+\nu_0)^2(4+\nu_0)^2(5+\nu_0)^2(1+2\nu_0)^2}\right]x^5 \\
&- \sigma_1 x^{\nu_0}\;[\text{etc., etc., }\cdots]
\end{aligned} \tag{11.122}$$

Equation (11.122) contains all the essentials of the second solution. However, in order to convert it into proper form, one needs to do two simple things.

(i) : Replace the arbitrary constant σ_1 in (11.122) by an arbitrary constant σ_2 everywhere.

(ii) : Knowing that the double root of the indicial equation for (11.99)-(A) is $\nu_0 = 0$, everywhere in (11.122), replace the symbol ν_0 with its value 0.

Once these changes have been executed, the numerical form of the second solution, $Y_{A;2}(x, 0)$, of differential equation (11.99)-(A) can be written as

$$\left(\frac{Y_{A;2}(x,0)}{\sigma_2}\right)_{Piaggio} = \exp(3\,x)\cdot\log(x) - 3\,x - \left(\frac{27}{4}\right)x^2 - \left(\frac{33}{4}\right)x^3 - \left(\frac{225}{32}\right)x^4 - \left(\frac{3,699}{800}\right)x^5 - \left(\frac{3,969}{1600}\right)x^6 - \left(\frac{88,209}{78,400}\right)x^7 - \left(\frac{554,769}{1,254,400}\right)x^8 - O(x^9) \quad . \tag{11.123}$$

Note: In going from (11.122) to (11.123), we have used the result, $G_A(x, \nu_0 = 0) = Y_{A;1}(x, 0) = \sigma_2\ \exp(3x)$, for the first solution that was given in (11.107). Note also that the arbitrary constant σ_1 has been replaced by arbitrary constant σ_2.

11.7.2 Method of Order Reduction

If a nonzero first solution, $y_1(x)$, of a second-order homogeneous linear ordinary differential equation is already known, a linearly independent second solution can be determined by a procedure that was studied in detail in a previous chapter in relation to (7.31)–(7.45). That procedure consists in representing the second solution $y_2(x)$ as a product of the first solution $y_1(x)$ and an unknown new function $f(x)$.

$$y_2(x) = f(x)\,y_1(x) \quad . \tag{11.124}$$

In this regard, (7.31)–(7.45) can be used without alteration as long as the old notation is aligned with the current notation. To that purpose, consider first the differential equation (7.31) presented below as (11.125)

$$\left[a_2(x)\,D^2 + a_1(x)\,D + a_0(x)\right]u(x) \;=\; 0 \quad , \tag{11.125}$$

and compare it to the current differential equation (11.126).

$$y''(x) + \left(\frac{1-3\,x}{x}\right)y'(x) - \left(\frac{3}{x}\right)y(x) \;=\; 0\,. \tag{11.126}$$

By comparing (11.125) and (11.126), we get

$$a_2(x) \;=\; 1 \;;\; a_1(x) \;=\; \left(\frac{1-3x}{x}\right) \;;\; a_0(x) \;=\; -\left(\frac{3}{x}\right) \tag{11.127}$$

According to (7.45), $y_2(x)$ is related to $y_1(x)$ as in (11.128).

$$\frac{y_2(x)}{y_1(x)} \;=\; f(x) \;=\; \sigma_3 \cdot \int \frac{\exp\left[-\int \frac{a_1(x)}{a_2(x)}\mathrm{d}x\right]}{[y_1(x)]^2}\,\mathrm{d}x\,. \tag{11.128}$$

Equation (11.128) involves two integrations. First the integral

$$-\int \frac{a_1(x)}{a_2(x)}\mathrm{d}x \;=\; -\int \left[\frac{1-3x}{x}\right]\mathrm{d}x \;=\; -\log(x) + 3\,x\,. \tag{11.129}$$

As a result, (11.128) can be written as

$$y_2(x) \;=\; \sigma_3\, y_1(x) \cdot \int \frac{\exp(3\,x)}{x\,[y_1(x)]^2}\,\mathrm{d}x. \tag{11.130}$$

Next, using the fact that $y_1(x) = \text{const.}\ \exp(3\,x)$, we have

$$y_2(x) \;=\; \sigma_4 \exp(3x) \int \frac{\exp(-3x)}{x}\mathrm{d}x\,. \tag{11.131}$$

The integral in (11.131) is readily done.

$$\begin{aligned}
&\int \frac{\exp(-3x)}{x}\mathrm{d}x \\
&= \int \left(\frac{1}{x} - 3 + \frac{9}{2}x - \frac{9}{2}x^2 + \frac{27}{8}x^3 - \frac{81}{40}x^4 + \frac{81}{80}x^5 - \frac{243}{560}x^6 \ldots\right)\mathrm{d}x \\
&= \log(x) - 3x + \frac{9}{4}x^2 - \frac{3}{2}x^3 + \frac{27}{32}x^4 - \frac{81}{200}x^5 + \frac{27}{160}x^6 - \frac{243}{3920}x^7 \ldots
\end{aligned}$$

As a result, (11.131) becomes

$$\begin{aligned}
\left(\frac{y_2(x)}{\sigma_4}\right) \;&=\; \exp(3x)\left[\log(x)\right] \\
&+ \exp(3x)\left[-3x + \frac{9}{4}x^2 - \frac{3}{2}x^3 + \frac{27}{32}x^4 - \frac{81}{200}x^5 + O(x^6)\right]
\end{aligned}$$

Finally, we need the power expansion of $\exp(3x)$.

$$\exp(3x) = 1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \frac{27}{8}x^4 + \frac{81}{40}x^5 + \frac{81}{80}x^6 + \frac{243}{560}x^7 + \frac{729}{4,480}x^8$$

This leads to

$$\begin{aligned}\left(\frac{y_2(x)}{\sigma_4}\right) &= \exp(3x)\left[\log(x)\right] \\ &+ \left(1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \frac{27}{8}x^4 + \text{etc.}\right).\left[-3x + \frac{9}{4}x^2 - \frac{3}{2}x^3 + \frac{27}{32}x^4 - \text{etc.}\right]\end{aligned}$$

As a result, the numerical representation of the second solution of differential equation (11.99)-(A), calculated by the method of order reduction, is the following.

$$\begin{aligned}&\left(\frac{Y_{A;2}(x,0)}{\sigma_4}\right)_{\text{ord.reduction}} \\ &= \exp(3\,x)\cdot\log(x) - 3\,x - \left(\frac{27}{4}\right)x^2 - \left(\frac{33}{4}\right)x^3 \\ &- \left(\frac{225}{32}\right)x^4 - \left(\frac{3,699}{800}\right)x^5 - \left(\frac{3,969}{1600}\right)x^6 \\ &- \left(\frac{88,209}{78,400}\right)x^7 - \left(\frac{554,769}{1,254,400}\right)x^8 \\ &- -\left(\frac{192,483}{1,254,400}\right)x^9 - \left(\frac{597,861}{12,544,000}\right)x^{10}\ ,\ \text{etc.}\end{aligned} \tag{11.132}$$

Note: $\left(\frac{Y_{A;2}(x,0)}{\sigma_4}\right)_{\text{ord.reduction}}$ in (11.132) is the same as $\left(\frac{Y_{A;2}(x,0)}{\sigma_4}\right)_{\text{Piaggio}}$ in (11.123). As usual, σ_0, σ_1, σ_2, etc., are arbitrary constants.

11.7.3 Complete Solution of (11.99)-(A)

Complete numerical solution of (11.99)-(A), $Y_{\text{A;Complete}}(x,0)$, consists of $Y_{A;1}(x,0)$ recorded in (11.107) and $Y_{A;2}(x,0)$ in (11.132). As such, (11.133)—which is the sum of these two—represents the complete solution of (11.99)-(A).

$$\begin{aligned}Y_{A;0}(x,0) &= Y_{A;1}(x,0) + Y_{A;2}(x,0) \\ &= \sigma_1\ \exp(3x) \\ &+ \sigma_2[\exp(3\,x)\cdot\log(x) - 3\,x - \left(\frac{27}{4}\right)x^2 - \left(\frac{33}{4}\right)x^3 \\ &- \left(\frac{225}{32}\right)x^4 - \left(\frac{3,699}{800}\right)x^5 - \left(\frac{3,969}{1600}\right)x^6\end{aligned}$$

$$- \left(\frac{88,209}{78,400}\right) x^7 - \left(\frac{554,769}{1,254,400}\right) x^8$$
$$- - \left(\frac{192,483}{1,254,400}\right) x^9 - \left(\frac{597,861}{12,544,000}\right) x^{10}] \ \ldots \ . \tag{11.133}$$

11.7.4 General Solution of (11.99)-(G)

For (11.99)-(G) the prescribed values of the variables are: $\beta = 1, \gamma = -1, \nu = 1$ and $\rho = 1$. As such the form of differential equation (11.98) that is being considered here is (11.134).

$$y''(x) + \left(\frac{x-1}{x}\right) y'(x) + \left(\frac{x+1}{x^2}\right) y(x) \ = \ 0 \, . \tag{11.134}$$

The relevant indicial equation,

$$\nu_0^2 + \nu_0(\gamma - 1) + \rho \ = \ \nu_0^2 - 2\nu_0 + 1 \ = \ 0 \ ,$$

has a double root: $\nu_1 = \nu_0 = 1$ and $\nu_2 = \nu_0 = 1$. As before, we work directly with the variable ν_0 and do not specify its numerical value. As such, (11.85), that relates a_1 to a_0, is written as

$$a_1 = -a_0 \left(\frac{\nu + \beta\,\nu_0}{\gamma + 2\,\nu_0}\right) \ = \ - a_0 \left(\frac{1 + \nu_0}{2\,\nu_0 - 1}\right) \ . \tag{11.135}$$

[Also see (11.100).] To calculate a_2 and higher-order a_n, rewrite (11.90) [Also see (11.101)].

$$\begin{aligned} a_n &= -a_{n-1} \left[\frac{\beta\,(n-1+\nu_0) + \nu}{(n+\nu_0)(n+\nu_0+\gamma-1) + \rho}\right] \\ &= -a_{n-1} \left[\frac{(n-1+\nu_0) + 1}{(n+\nu_0)(n+\nu_0-1-1) + 1}\right] \\ &= -a_{n-1} \left[\frac{(n+\nu_0)}{(n+\nu_0-1)^2}\right] \end{aligned} \tag{11.136}$$

For $n = 2$, (11.135) and (11.136) give

$$a_2 \ = \ \left[\frac{(2+\nu_0)}{(\nu_0+1)(2\,\nu_0-1)}\right] a_0 \ . \tag{11.137}$$

For $n = 3$, (11.135) and (11.136) give

$$a_3 = -\left[\frac{(3+\nu_0)}{(\nu_0+2)(\nu_0+1)(2\,\nu_0-1)}\right]a_0 \ . \tag{11.138}$$

For $n = 4$, (11.135) and (11.136) give

$$a_4 = \left[\frac{(4+\nu_0)}{(\nu_0+3)(\nu_0+2)(\nu_0+1)(2\,\nu_0-1)}\right]a_0 \ . \tag{11.139}$$

For $n = 5$, (11.135) and (11.136) give

$$a_5 = -\left[\frac{(5+\nu_0)}{(\nu_0+4)(\nu_0+3)(\nu_0+2)(\nu_0+1)(2\,\nu_0-1)}\right]a_0 \ . \tag{11.140}$$

For $n = 6, 7, 8$, (11.135) and (11.136) give

$$\begin{aligned} a_6 &= \left[\frac{(6+\nu_0)}{(\nu_0+5)(\nu_0+4)(\nu_0+3)(\nu_0+2)(\nu_0+1)(2\,\nu_0-1)}\right]a_0 \\ a_7 &= -\left[\frac{(7+\nu_0)}{(\nu_0+6)(\nu_0+5)(\nu_0+4)(\nu_0+3)(\nu_0+2)(\nu_0+1)(2\,\nu_0-1)}\right]a_0 \\ a_8 &= \left[\frac{(8+\nu_0)}{(\nu_0+7)(\nu_0+6)(\nu_0+5)(\nu_0+4)(\nu_0+3)(\nu_0+2)(\nu_0+1)(2\,\nu_0-1)}\right]a_0. \end{aligned} \tag{11.141}$$

The above results—that is (11.135)–(11.141)—are listed below.

$$\begin{aligned} &\left(\frac{G_G(x,\nu_0)}{\sigma_0\,\nu_0\,x^{\nu_0}}\right) \\ &= 1 - \left(\frac{1+\nu_0}{2\,\nu_0-1}\right)x + \left[\frac{(2+\nu_0)}{(\nu_0+1)(2\,\nu_0-1)}\right]x^2 \\ &- \left[\frac{(3+\nu_0)}{(\nu_0+2)(\nu_0+1)(2\,\nu_0-1)}\right]x^3 + \left[\frac{(4+\nu_0)}{(\nu_0+3)(\nu_0+2)(\nu_0+1)(2\,\nu_0-1)}\right]x^4 \\ &- \left[\frac{(5+\nu_0)}{(\nu_0+4)(\nu_0+3)(\nu_0+2)(\nu_0+1)(2\,\nu_0-1)}\right]x^5 \\ &+ \left[\frac{(6+\nu_0)}{(\nu_0+5)(\nu_0+4)(\nu_0+3)(\nu_0+2)(\nu_0+1)(2\,\nu_0-1)}\right]x^6 \\ &- \left[\frac{(7+\nu_0)}{(\nu_0+6)(\nu_0+5)(\nu_0+4)(\nu_0+3)(\nu_0+2)(\nu_0+1)(2\,\nu_0-1)}\right]x^7 \\ &+ \left[\frac{(8+\nu_0)}{(\nu_0+7)(\nu_0+6)(\nu_0+5)(\nu_0+4)(\nu_0+3)(\nu_0+2)(\nu_0+1)(2\,\nu_0-1)}\right]x^8 \\ &- O(x^9) \end{aligned} \tag{11.142}$$

$G_G(x, \nu_0)$ in (11.142) is the general solution of (11.99)-(G).

11.7.5 First Solution of (11.99)-(G)

For (11.99)-(G), $\beta = 1, \gamma = -1, \nu = 1$ and $\rho = 1$ and the relevant differential equation is (11.134). The double root of the indicial equation is $\nu_0 = 1$. As stated before, the first solution, $Y_{G;1}(x, \nu_0)$, is the general solution, $G_G(x, \nu_0)$, itself while the second solution, $Y_{G;2}(x, \nu_0)$, is the differential with respect to ν_0 of the general solution. And at the end of the process, one converts the first and the second solution into numerical format. This is done by replacing the symbol ν_0 by its actual value: which is the double root of the relevant indicial equation. The notation to be used is the following.

For unspecified value of ν_0 the first and the second solutions of (11.99)-(G) are denoted as $Y_{G;1}(x, \nu_0)$ and $Y_{G;2}(x, \nu_0)$. And furthermore, as stated earlier,

$$Y_{G;1}(x, \nu_0) \;=\; G_G(x, \nu_0) \quad , \tag{11.143}$$

and

$$Y_{G;2}(x, \nu_0) \;=\; \left[\frac{d\, Y_{G;1}(x, \nu_0)}{d\, \nu_0}\right] \;=\; \left[\frac{d\, G_G(x, \nu_0)}{d\, \nu_0}\right] \quad . \tag{11.144}$$

For specific value of ν_0 equal to the double root of the indicial equation—which for the present case is $\nu_0 = 1$—the first and the second solutions of (11.99)-(G) are to be denoted as

$$G_G(x, \nu_0 = 1) \;=\; Y_{G;1}(x, \nu_0 = 1) \;\equiv\; Y_{G;1}(x, 1) \quad , \tag{11.145}$$

and

$$Y_{G;2}(x, \nu_0 = 1) \;=\; Y_{G;2}(x, 1) \;=\; \left[\frac{d\, Y_{G;1}(x, \nu_0)}{d\, \nu_0}\right]_{\nu_0 = 1} \quad . \tag{11.146}$$

Setting $\nu_0 = 1$ everywhere in the general solution $G_G(x, \nu_0)$, that was given in (11.142), leads to the following result for the first solution, $Y_{G;1}(x, 1)$, of (11.99)-(G). [Note: Compare (11.113).]

$$\begin{aligned}(G) \;:\; \frac{Y_{G;1}(x, 1)}{\sigma_1 x} &= \frac{G_G(x, \nu_0 = 1)}{\sigma_1 x} \;=\; (1 - x)\exp(-x) \\ &= \; 1 - 2\,x + \frac{3\,x^2}{2} - \frac{2\,x^3}{3} + \frac{5\,x^4}{24} - \frac{x^5}{20} + \frac{7\,x^6}{720} - \frac{x^7}{630} + \cdots \quad .\end{aligned} \tag{11.147}$$

11.7.6 Second Solution of (11.99)-(G)

To save space, only the Piaggio-like second solution is being presented.

11.7.7 Piaggio-Like Second Solution

As stated above and expressed in (11.144), for unspecified values of ν_0, the second solution, $Y_{G;2}(x, \nu_0)$, of (11.99)-(G) is the differential, with respect to the variable ν_0, of its general solution given in (11.142). That is

$$
\begin{aligned}
Y_{G;2}(x, \nu_0) \;&=\; \left[\frac{d\, G_G(x, \nu_0)}{d\, \nu_0}\right] \\
&= G_G(x, \nu_0) \cdot \log(x) + \sigma_1\, x^{\nu_0} \left[1 + \frac{(1 + 2\nu_0 - 2\nu_0^2)}{(2\,\nu_0 - 1)^2} x\right] \\
&- \sigma_1\, x^{\nu_0} \left[\frac{2 + 2\nu_0 + 3{\nu_0}^2}{(2\nu_0^2 + \nu_0 - 1)^2}\right] x^2 \\
&+ \sigma_1 x^{\nu_0} \left[2\frac{(3 + 2\nu_0 + 7{\nu_0}^2 + 6{\nu_0}^3 + {\nu_0}^4)}{(2\nu_0^3 + 5\nu_0^2 + \nu_0 - 2)^2}\right] x^3 \\
&- \sigma_1 x^{\nu_0} \left[\frac{24 + 12\nu_0 + 63{\nu_0}^2 + 88{\nu_0}^3 + 35\nu_0^4 + 4\nu_0^5}{(2\nu_0^4 + 11\nu_0^3 + 16\nu_0^2 + \nu_0 - 6)^2}\right] x^4 \\
&+ \sigma_1 x^{\nu_0} \left[3\frac{(2\nu_0^6 + 26\nu_0^5 + 115\nu_0^4 + 200\nu_0^3 + 109\nu_0^2 + 16\nu_0 + 40}{[(\nu_0 + 4)(\nu_0 + 3)(\nu_0 + 2)(\nu_0 + 1)(2\nu_0 - 1)]^2}\right] x^5 \\
&- O(x^{\nu_0}\, x^6)
\end{aligned}
\tag{11.148}
$$

Equation (11.148) contains all the essentials of the second solution. However, in order to convert it into proper form, one needs to do two simple things.

(i) : Replace the arbitrary constant σ_1 in (11.148) by an arbitrary constant σ_2 everywhere.

(ii) : Knowing that the double root of the indicial equation for (11.99)-(G) is $\nu_0 = 1$, everywhere in (11.148), replace the symbol ν_0 with its value 1.

Once these changes have been introduced the second solution, $Y_{G;2}(x, \nu_0 = 1) \rightarrow Y_{G;2}(x)$, of (11.99)-(G) can be written in its final form.

$$
\begin{aligned}
\left(\frac{Y_{G;\,2}(x, \nu_0 = 1)}{\sigma_2}\right)_{Piaggio} \;&=\; \left(\frac{Y_{G;\,2}(x)}{\sigma_2}\right)_{Piaggio} \\
&= x(1 - x)\exp(-x)\log(x) + x + x^2 - \left(\frac{7}{4}\right)x^3 + \left(\frac{19}{18}\right)x^4 - \left(\frac{113}{288}\right)x^5 \\
&+ \left(\frac{127}{1,\,200}\right)x^6 - O(x^7)
\end{aligned}
\tag{11.149}
$$

Note: In going from (11.148) to (11.149), we have used the result, $G_G(x, \nu_0 = 1) = Y_{G;1}(x, \nu_0 = 1) = Y_{G;1}(x) = \sigma_1\, x(1-x)\exp(-x)$ for the first solution that was given in (11.147). Of course, the arbitrary constant σ_1 there has been changed to σ_2 here.

Exercise for the Reader: (11.99)-(B)

Follow the above procedure and find the second solution for differential equation (11.99)-(B).

11.8 Examples Group V

Consider the differential equation.

$$y''(x) + \left(\frac{\beta x + \gamma}{x}\right) y'(x) + \left(\frac{\nu x + \rho}{x^2}\right) y(x) \;=\; 0 \;. \tag{11.150}$$

The four parameters β, γ, ν, ρ referred to in (11.150) are provided in (11.151).

$$\begin{aligned}
&(1) \;: [\beta = 1\,;\; \gamma = -2\,;\; \nu = 1\,;\; \rho = 2\,] \;. \\
&(2) \;: [\beta = 1\,;\; \gamma = -3\,;\; \nu = 1\,;\; \rho = 3\,] \;. \\
&(3) \;: [\beta = 1\,;\; \gamma = -4\,;\; \nu = 1\,;\; \rho = 4\,] \;. \\
&(4) \;: [\beta = 1\,;\; \gamma = -5\,;\; \nu = 1\,;\; \rho = 5\,] \;. \\
&(5) \;: [\beta = 1\,;\; \gamma = -6\,;\; \nu = 1\,;\; \rho = 6\,] \;.
\end{aligned} \tag{11.151}$$

Use (11.85); set $n = 2, 3, 4, 5, 6, 7$ in (11.86) and remember that the first solution of these differential equations makes use of the larger of the two roots of the indicial equation: that is $\nu_0 = \nu_1$. The smaller root, $\nu_0 = \nu_2$, of the indicial equation is used for the second solution. Because of space constraints, provide detailed information about the general solution, first solution, and two different techniques for working out the second solution, only for (11.151)-(4).

11.8.1 General Solution of (11.151)-(4)

For (11.151)-(4)—that is of the form of (11.150)—the prescribed values of the variables are: $\beta = 1, \gamma = -5, \nu = 1$ and $\rho = 5$. Accordingly, the relevant differential equation to consider is

$$y''(x) + \left(\frac{x-5}{x}\right) y'(x) + \left(\frac{x+5}{x^2}\right) y(x) \;=\; 0 \;. \tag{11.152}$$

The two roots of its indicial equation,

$$\nu_0^2 + \nu_0(\gamma - 1) + \rho \;=\; \nu_0^2 - 6\nu_0 + 5 \;=\; 0 \;,$$

are $\nu_0 = 5$ and $\nu_0 = \;1$. Work directly with the variable ν_0 and not specify either of its two roots. As such, (11.85), that relates a_1 to a_0, is written as

$$a_1 = -a_0 \left(\frac{\nu + \beta\,\nu_0}{\gamma + 2\,\nu_0} \right) \;=\; -\left(\frac{1 + \nu_0}{2\,\nu_0 - 5} \right) a_0 \;. \tag{11.153}$$

[Note: Compare (11.100).] To calculate a_2 and higher-order a_n, rewrite (11.90) [Note: See (11.100).]

$$\begin{aligned} a_n &= -a_{n-1} \left[\frac{\beta\,(n - 1 + \nu_0) + \nu}{(n + \nu_0)(n + \nu_0 + \gamma - 1) + \rho} \right] \\ &= -a_{n-1} \left[\frac{(n - 1 + \nu_0) + 1}{(n + \nu_0)(n + \nu_0 - 5 - 1) + 5} \right] \\ &= -a_{n-1} \left[\frac{(n + \nu_0)}{(n + \nu_0 - 5)(n + \nu_0 - 1)} \right] \;. \end{aligned} \tag{11.154}$$

For $n = 2$, (11.153) and (11.154) give

$$a_2 \;=\; \left[\frac{(2 + \nu_0)}{(\nu_0 - 3)(2\,\nu_0 - 5)} \right] a_0 \;. \tag{11.155}$$

For $n = 3$, (11.153) and (11.154) give

$$a_3 \;=\; -\left[\frac{(3 + \nu_0)}{(\nu_0 - 3)(\nu_0 - 2)(2\,\nu_0 - 5)} \right] a_0 \;. \tag{11.156}$$

For $n = 4$, (11.153) and (11.154) give

$$a_4 \;=\; \left[\frac{(4 + \nu_0)}{(\nu_0 - 3)(\nu_0 - 2)(\nu_0 - 1)(2\,\nu_0 - 5)} \right] a_0 \;. \tag{11.157}$$

For $n = 5$, (11.153) and (11.154) give

$$a_5 = -\left[\frac{(5 + \nu_0)}{(\nu_0 - 3)(\nu_0 - 2)(\nu_0 - 1)(\nu_0)(2\,\nu_0 - 5)} \right] a_0 \;. \tag{11.158}$$

For $n = 6, 7, 8$, (11.153) and (11.154) give

$$\begin{aligned} a_6 &= \left[\frac{(6+\nu_0)}{(\nu_0-3)(\nu_0-2)(\nu_0-1)(\nu_0)(\nu_0+1)(2\,\nu_0-5)}\right] a_0 \\ a_7 &= -\left[\frac{(7+\nu_0)}{(\nu_0-3)(\nu_0-2)(\nu_0-1)(\nu_0)(\nu_0+1)(\nu_0+2)(2\,\nu_0-5)}\right] a_0 \\ a_8 &= \left[\frac{(8+\nu_0)}{(\nu_0-3)(\nu_0-2)(\nu_0-1)(\nu_0)(\nu_0+1)(\nu_0+2)(\nu_0+3)(2\,\nu_0-5)}\right] a_0 \ . \end{aligned} \tag{11.159}$$

These results are listed below.

$$\begin{aligned} &\left(\frac{G(x,\nu_0)}{\sigma_0\, x^{\nu_0}}\right) \\ &= 1 - \left(\frac{1+\nu_0}{2\,\nu_0-5}\right) x + \left[\frac{(2+\nu_0)}{(\nu_0-3)(2\,\nu_0-5)}\right] x^2 \\ &- \left[\frac{(3+\nu_0)}{(\nu_0-3)(\nu_0-2)(2\,\nu_0-5)}\right] x^3 + \left[\frac{(4+\nu_0)}{(\nu_0-3)(\nu_0-2)(\nu_0-1)(2\,\nu_0-5)}\right] x^4 \\ &- \left[\frac{(5+\nu_0)}{(\nu_0-3)(\nu_0-2)(\nu_0-1)(\nu_0)(2\,\nu_0-5)}\right] x^5 \\ &+ \left[\frac{(6+\nu_0)}{(\nu_0-3)(\nu_0-2)(\nu_0-1)(\nu_0)(\nu_0+1)(2\,\nu_0-5)}\right] x^6 \\ &- \left[\frac{(7+\nu_0)}{(\nu_0-3)(\nu_0-2)(\nu_0-1)(\nu_0)(\nu_0+1)(\nu_0+2)(2\,\nu_0-5)}\right] x^7 \\ &+ \left[\frac{(8+\nu_0)}{(\nu_0-3)(\nu_0-2)(\nu_0-1)(\nu_0)(\nu_0+1)(\nu_0+2)(\nu_0+3)(2\,\nu_0-5)}\right] x^8 \\ &- O(x^9) \end{aligned} \tag{11.160}$$

$G(x, \nu_0)$ in (11.160) is the general solution of (11.152) that represents (11.151)-(4).

11.8.2 First Solution of (11.151)-(1)–(5)

Consider differential equation (11.150) and set the values of β, γ, ν, ρ as provided in (11.151)-(1)–(5). For differential equation (11.151)-(1), $\beta = 1, \gamma = -2, \nu = 1$ and $\rho = 2$, and the two roots of the indicial equation,

$$\nu_0^2 + \nu_0(\gamma - 1) + \rho \ = \ 0 \ = \nu_0^2 - 3\nu_0 + 2 \ ,$$

are $\nu_0 = 2$ and $\nu_0 = 1$. Accordingly, setting $\nu_0 = 2$ everywhere in its general solution gives the first solution, $y_{1;1}(x, \nu_0 = 2)$, for (11.151)-(1). [Note, to save space, neither the general solution nor the first solution for (11.151)-(1), (2), (3) and (5) are recorded

here.]

$$(1): \left[\frac{y_{1;1}(x,\nu_0=2)}{\sigma_1 x^2}\right] = 1 - \frac{3x}{2} + x^2 - \frac{5x^3}{12} + \frac{x^4}{8} - \frac{7x^5}{240} + \frac{x^6}{180} - \frac{x^7}{1,120} + \cdots$$
$$= \left(1 - \frac{x}{2}\right)\exp(-x). \tag{11.161}$$

For differential equation (11.151)-(2), $\beta = 1$, $\gamma = -3$, $\nu = 1$ and $\rho = 3$, and the two roots of the indicial equation are $\nu_0 = 3$ and $\nu_0 = 1$. The following result obtains for its first solution, $y_{1;2}(x, \nu_0 = 3)$.

$$(2): \left[\frac{y_{1;2}(x,\nu_0=3)}{\sigma_1 x^3}\right] = 1 - \frac{4x}{3} + \frac{5x^2}{6} - \frac{x^3}{3} + \frac{7x^4}{72} - \frac{x^5}{45} + \frac{x^6}{240} - \frac{x^7}{1,512} + \cdots$$
$$= \left(1 - \frac{x}{3}\right)\exp(-x)\ . \tag{11.162}$$

For differential equation (11.151)-(3), $\beta = 1$, $\gamma = -4$, $\nu = 1$ and $\rho = 4$, and the two roots of the indicial equation are $\nu_0 = 4$ and $\nu_0 = 1$. We get

$$(3): \left[\frac{y_{1;3}(x,\nu_0=4)}{\sigma_1 x^4}\right] = 1 - \frac{5x}{4} + \frac{3x^2}{4} - \frac{7x^3}{24} + \frac{x^4}{12} - \frac{3x^5}{160} + \frac{x^6}{288} - \frac{11x^7}{20,160} + \cdots$$
$$= \left(1 - \frac{x}{4}\right)\exp(-x)\ . \tag{11.163}$$

For differential equation (11.151)-(4), $\beta = 1$, $\gamma = -5$, $\nu = 1$ and $\rho = 5$, and the two roots of the indicial equation are $\nu_0 = 5$ and $\nu_0 = 1$. Therefore,

$$(4): \left[\frac{y_{1;4}(x,\nu_0=5)}{\sigma_1 x^5}\right] = 1 - \frac{6x}{5} + \frac{7x^2}{10} - \frac{4x^3}{15} + \frac{3x^4}{40} - \frac{x^5}{60} + \frac{11x^6}{3,600} - \frac{x^7}{2,100} + \cdots$$
$$= \left(1 - \frac{x}{5}\right)\exp(-x)\ . \tag{11.164}$$

And finally for differential equation (11.151)-(5), we have $\beta = 1$, $\gamma = -6$, $\nu = 1$ and $\rho = 6$, and the two roots of the indicial equation are $\nu_0 = 6$ and $\nu_0 = 1$. For root $\nu_0 = 6$, the result for the first solution is

$$(5): \left[\frac{y_{1;5}(x,\nu_0=6)}{\sigma_1 x^6}\right] = 1 - \frac{7x}{6} + \frac{2x^2}{3} - \frac{x^3}{4} + \frac{5x^4}{72} - \frac{11x^5}{720} + \frac{x^6}{360} - \frac{13x^7}{30,240} + \cdots$$
$$= \left(1 - \frac{x}{6}\right)\exp(-x)\ . \tag{11.165}$$

11.9 Equation (11.151)-(4)

11.9.1 Second Solutions

Use the prescribed values of the variables noted in (11.151)-(4). That is $\beta = 1, \gamma = -5, \nu = 1$ and $\rho = 5$. Insert them into differential equation (11.150). This leads to (11.166) which is the relevant differential equation for (11.151)-(4).

$$y''(x) + \left(\frac{x-5}{x}\right) y'(x) + \left(\frac{x+5}{x^2}\right) y(x) \;=\; 0 \;. \tag{11.166}$$

The two roots of its indicial equation,

$$\nu_0^2 + \nu_0(\gamma - 1) + \rho \;=\; \nu_0^2 - 6\nu_0 + 5 = \; 0 \;,$$

are $\nu_0 = 5$ and $\nu_0 = \; 1$.

The second solution uses the smaller of the two roots namely $\nu_0 = \; 1$. A cursory look at (11.151)-(4)'s general solution (11.160) shows how it is replete with infinities. Infinities arise from the term $(\nu_0 - 1)$ in the denominator for the reason that this term goes to zero when $\nu_0 = 1$. To obviate the occurrence of these infinities, one proceeds as follows.

11.9.2 Piaggio's Solution

Replace the arbitrary constant σ_0 in the general solution (11.160) by $\sigma_{\text{pia};4}(\nu_0 - 1)$ where $\sigma_{\text{pia};4}$ is also an arbitrary constant. Doing this transforms $G(x, \nu_0)$ into $G_{\text{pia}}(x, \nu_0)$. More importantly, it gets rid everywhere of the pesky denominator $(\nu_0 - 1)$. As a result, the general solution (11.160) changes into

$$\begin{aligned}
\left(\frac{G_{pia}(x,\nu_0)}{\sigma_{pia;4}\, x^{\nu_0}}\right) &= (\nu_0 - 1)\left\{1 - \left(\frac{1+\nu_0}{2\,\nu_0 - 5}\right) x\right\} \\
&+ (\nu_0 - 1)\left\{\left[\frac{(2+\nu_0)}{(\nu_0-3)(2\,\nu_0-5)}\right] x^2 \;-\left[\frac{(3+\nu_0)}{(\nu_0-3)(\nu_0-2)(2\,\nu_0-5)}\right] x^3\right\} \\
&+ \left[\frac{(4+\nu_0)}{(\nu_0-3)(\nu_0-2)(2\,\nu_0-5)}\right] x^4 - \left[\frac{(5+\nu_0)}{(\nu_0-3)(\nu_0-2)(\nu_0)(2\,\nu_0-5)}\right] x^5 \\
&+ \left[\frac{(6+\nu_0)}{(\nu_0-3)(\nu_0-2)(\nu_0)(\nu_0+1)(2\,\nu_0-5)}\right] x^6 \\
&- \left[\frac{(7+\nu_0)}{(\nu_0-3)(\nu_0-2)(\nu_0)(\nu_0+1)(\nu_0+2)(2\,\nu_0-5)}\right] x^7
\end{aligned}$$

$$+\left[\frac{(8+\nu_0)}{(\nu_0-3)(\nu_0-2)(\nu_0)(\nu_0+1)(\nu_0+2)(\nu_0+3)(2\,\nu_0-5)}\right]x^8$$
$$-\,O(x^9) \tag{11.167}$$

The next thing to do is to differentiate $G_{\text{pia}}(x,\nu_0)$ with respect to ν_0. And call the result $y_{2;4}(x,\nu_0)$.

$$\begin{aligned}
&\frac{dG_{pia}(x,\nu_0)}{d\nu_0} \;=\; y_{2;4}(x,\nu_0)\\
&=G_{pia}(x,\nu_0)\log(x)\\
&+\sigma_{pia;4}\,x^{\nu_0}\cdot\Bigg[1-2\left(\frac{\nu_0^2-5\nu_0+1}{(2\nu_0-5)^2}\right)x+\left(\frac{27}{(2\nu_0-5)^2}-\frac{10}{(\nu_0-3)^2}\right)x^2\\
&-\left(\frac{66}{(2\nu_0-5)^2}-\frac{12}{(\nu_0-3)^2}-\frac{5}{(\nu_0-2)^2}\right)x^3\\
&-\left(\frac{4\nu_0^3+9\nu_0^2-120\nu_0+178}{(2\nu_0^3-15\nu_0^2+37\nu_0-30)^2}\right)x^4\\
&+2\left(\frac{3\nu_0^4+5\nu_0^3-94\nu_0^2+185\nu_0-75}{\nu^2(2\nu_0^3-15\nu_0^2+37\nu_0-30)^2}\right)x^5\\
&-\left(\frac{8\nu_0^5+21\nu_0^4-268\nu_0^3+403\nu_0^2+84\nu_0-180}{\nu_0^2(2\nu_0^4-13\nu_0^3+22\nu_0^2+7\nu_0-30)}\right)x^6\\
&+\left(\frac{10\nu_0^6+48\nu_0^5-327\nu_0^4-10\nu_0^3+1055\nu_0^2-224\nu_0-420}{\nu_0^2(2\nu_0^5-9\nu_0^4-4\nu_0^3+51\nu_0^2-16\nu_0-60)^2}\right)x^7\\
&-O(x^8)\Bigg]
\end{aligned} \tag{11.168}$$

Set $\nu_0 = 1$ in both (11.167) and (11.168). Equation (11.167) gives

$$\begin{aligned}
&\left(\frac{G_{pia}(x,\nu_0=1)}{\sigma_{pia;4}}\right)\\
&=-\left(\frac{5x^5}{6}\right)\left[1-\left(\frac{6x}{5}\right)+\left(\frac{7x^2}{2!5}\right)-\left(\frac{8x^3}{3!5}\right)+\left(\frac{9x^4}{4!5}\right)-O(x^5)\right]\\
&=-\left(\frac{5}{6}\right)x^5\left(1-\frac{x}{5}\right)\exp(-x)\;\;,
\end{aligned} \tag{11.169}$$

and (11.168) yields

$$\begin{aligned}
&\left(\frac{y_{2;4}(x,\nu_0=1)}{\sigma_{pia;4}}\right)\\
&=\left[\frac{G_{pia}(x,\nu_0=1)\log(x)}{\sigma_{pia;4}}\right]+x+\left(\frac{2}{3}\right)x^2+\left(\frac{1}{2}\right)x^3+\left(\frac{2}{3}\right)x^4\\
&\quad-O(x^5).
\end{aligned} \tag{11.170}$$

Combining (11.169) and (11.170) leads to Piaggio's suggested second solution of (11.151)-(4).

$$\left(\frac{y_{2;4}(x,\nu_0=1)}{\sigma_{pia;4}}\right) \equiv \left(\frac{y_{2;4}(x,\nu_0=1)}{\sigma_{pia;4}}\right)_{\text{piaggio}}$$
$$= -\left(\frac{5}{6}\right)x^5\left(1-\frac{x}{5}\right)\exp(-x)\,.\,\log(x)+x+\left(\frac{2}{3}\right)x^2+\left(\frac{1}{2}\right)x^3+\left(\frac{2}{3}\right)x^4$$
$$-O(x^5). \tag{11.171}$$

11.10 Solution by Method of Order Reduction

As explained earlier, given the availability of a nonzero solution, $y_1(x)$, of a second-order homogeneous linear ordinary differential equation a linearly independent second solution can be determined by a procedure known as order reduction. [Compare (7.31)–(7.45).] This procedure consists in representing the second solution, $y_2(x)$, as a product of the first solution, which currently is $y_1(x) = \sigma_1 x^5\left(1-\frac{x}{5}\right)\exp(-x)$, and an unknown new function $f(x)$.

$$y_2(x) = f(x)\,y_1(x) \quad . \tag{11.172}$$

For a differential equation

$$\left[a_2(x)\,D^2+a_1(x)\,D+a_0(x)\right]u(x) \;=\; 0 \tag{11.173}$$

the second solution, $y_2(x)$, is given by the relationship

$$y_2(x) = \; y_1(x)\;f(x) \;=\; \sigma_0\,y_1(x)\cdot\int\frac{\exp\left[-\int\frac{a_1(x)}{a_2(x)}\mathrm{d}x\right]}{[y_1(x)]^2}\,\mathrm{d}x\,. \tag{11.174}$$

Note, σ_0 is an arbitrary constant. By comparing (11.173) and the current differential equation,

$$y''(x)+y'(x)\left(\frac{x-5}{x}\right)+\left(\frac{x+5}{x^2}\right)y(x) \;=\; 0\;, \tag{11.175}$$

we get

$$a_2(x) \;=\; 1\;;\;\; a_1(x) \;=\; \left(\frac{x-5}{x}\right)\;;\;\; a_0(x) \;=\; \left(\frac{x+5}{x}\right)\;. \tag{11.176}$$

The inside integral in (11.174) needs to be done first.

$$\exp\left[-\int \frac{a_1(x)}{a_2(x)}\mathrm{d}x\right] = \exp\left[-\int \frac{x-5}{x}\mathrm{d}x\right] = \exp\left[\int \frac{5}{x}\mathrm{d}x - \int \mathrm{d}x\right]$$
$$= \exp\left[5\log(x) - x\right] = x^5 \exp(-x)\ . \tag{11.177}$$

Use (11.177); insert $y_1(x) = \sigma_1 x^5 \left(1 - \frac{x}{5}\right)\exp(-x)$ in (11.174); and write the resultant of (11.174) as

$$\frac{y_2(x)}{(-4\sigma_2)} = x^5 \left(1 - \frac{x}{5}\right)\exp(-x)\left[\int \frac{x^5 \exp(-x)}{\left\{x^5 \exp(-x)(1 - \frac{x}{5})\right\}^2}\,\mathrm{d}x\right]$$
$$= x^5 \left(1 - \frac{x}{5}\right)\exp(-x)\left[\int \frac{\exp(x)}{x^5 \left(1 - \frac{x}{5}\right)^2}\,\mathrm{d}x\right]\ . \tag{11.178}$$

where $(-4\sigma_2)$ is a new arbitrary constant. Power expand the integrand,

$$\frac{\exp(x)}{x^5 \left(1 - \frac{x}{5}\right)^2} = \frac{1}{x^5} + \frac{7}{5x^4} + \frac{51}{50x^3} + \frac{389}{750x^2} + \frac{5}{24x} + (x^3)\ , \tag{11.179}$$

and integrate (11.179).

$$\int \frac{\exp(x)}{x^5 \left(1 - \frac{x}{5}\right)^2}\mathrm{d}x = -\left(\frac{1}{4}\right)x^{-4} - \left(\frac{7}{15}\right)x^{-3} - \left(\frac{51}{100}\right)x^{-2} - \left(\frac{389}{750}\right)x^{-1}$$
$$+ \left(\frac{5}{24}\right)\log(x) + O(x^2)\ . \tag{11.180}$$

Power expand $x^5 \left(1 - \frac{x}{5}\right)\exp(-x)$.

$$x^5 \left(1 - \frac{x}{5}\right)\exp(-x) = x^5 - \left(\frac{6}{5}\right)x^6 + \left(\frac{7}{10}\right)x^7 - \left(\frac{4}{15}\right)x^8$$
$$+ O(x^9) \tag{11.181}$$

The right-hand side of (11.178) is equal to the product of the left-hand sides of (11.181) and (11.180). Below we equate that with the product of the right-hand sides of (11.180) and (11.181). We get

$$\frac{y_2(x)}{(-4\sigma_2)} = x^5 \left(1 - \frac{x}{5}\right)\exp(-x)\int \frac{\exp(x)}{x^5 \left(1 - \frac{x}{5}\right)^2}\mathrm{d}x$$
$$= x^5 \left(1 - \frac{x}{5}\right)\exp(-x)\left[-\left(\frac{1}{4}\right)x^{-4} - \left(\frac{7}{15}\right)x^{-3} - \left(\frac{51}{100}\right)x^{-2} - \left(\frac{389}{750}\right)x^{-1}\right]$$
$$+ x^5 \left(1 - \frac{x}{5}\right)\exp(-x)\left[\left(\frac{5}{24}\right)\log(x) + O(x^2)\right]\ ,$$

$$
\begin{aligned}
&= \left[x^5 - \left(\frac{6}{5}\right)x^6 + \left(\frac{7}{10}\right)x^7\right]\left[-\left(\frac{1}{4}\right)x^{-4} - \left(\frac{7}{15}\right)x^{-3} - \left(\frac{51}{100}\right)x^{-2}\right] \\
&+ x^5\left(1 - \frac{x}{5}\right)\exp(-x)\left[\left(\frac{5}{24}\right)\log(x) + O(x^2)\right] . \\
&= \left(\frac{-1}{4}\right)\left\{\left(\frac{-5}{6}\right)x^5\left(1 - \frac{x}{5}\right)\exp(-x).\log(x) + x + \frac{2x^2}{3} + \frac{x^3}{2} + \frac{2x^4}{3} + O(x^5)\right\}.
\end{aligned}
$$

Finally, multiply both sides by -4 to get the second solution of (11.151)-(4) by the method of order reduction.

$$
\begin{aligned}
\frac{y_2(x)}{\sigma_2} &\equiv \left(\frac{y_{2;4}(x, \nu_0 = 1)}{\sigma_{pia;4}}\right)_{ord.reduc.} \\
&= \left\{\left(\frac{-5}{6}\right)x^5\left(1 - \frac{x}{5}\right)\exp(-x).\log(x) + x + \frac{2x^2}{3} + \frac{x^3}{2} + \frac{2x^4}{3} + O(x^5)\right\} .
\end{aligned}
\tag{11.182}
$$

Comment: Equation (11.182)—meaning $\left(\frac{y_{2;4}(x,\nu_0=1)}{\sigma_{\text{pia};4}}\right)_{\text{ord.reduc.}}$—agrees with Piaggio-like second solution, $\left(\frac{y_{2;4}(x,\nu_0=1)}{\sigma_{\text{pia};4}}\right)_{\text{piaggio}}$, of (11.151)-(4) recorded in (11.171).

11.10.1 Complete Solution

Complete solution, $y_{\text{complete};4}(x)$, of differential equation (11.151)-(4) is the sum of the first- and the second solutions given in (11.164) and (11.182).

$$
\begin{aligned}
&y_{complete;4}(x) \\
&\sigma_1 x^5\left(1 - \frac{x}{5}\right)\exp(-x) \\
&+ \sigma_2\left\{\left(\frac{-5}{6}\right)x^5\left(1 - \frac{x}{5}\right)\exp(-x).\log(x) + x + \frac{2x^2}{3} + \frac{x^3}{2} + \frac{2x^4}{3} + O(x^5)\right\} .
\end{aligned}
\tag{11.183}
$$

11.11 Bessel's Equation of Order Zero

Indicial Equation Roots Are Equal

Consider (11.184).

$$y''(x) + \left(\frac{\beta x + \gamma}{x}\right) y'(x) + \left(\frac{\mu x^2 + \nu x + \rho}{x^2}\right) y(x) \;=\; 0 \;. \qquad (11.184)$$

Bessel's equation of order zero is obtained by setting

$\beta = 0, \gamma = 1, \mu = 1, \nu = 0$ and $\rho = 0$ in (11.184).

$$y''(x) + \left(\frac{1}{x}\right) y'(x) + y(x) \;=\; 0 \;. \qquad (11.185)$$

It has a regular singular point at $x = 0$, and its indicial equation has a double root $\nu_0 = 0$. Accordingly, it is a member of category 2. As such the following rule applies.

If the two roots—ν_1 and ν_2—of the indicial equation are equal—meaning if $\nu_1 = \nu_2 \equiv \nu_0$, or equivalently $\rho = \left(\frac{\gamma-1}{2}\right)^2$, and the relevant differential equation—meaning (11.185)—has a regular singular point at $x = x_0$, then it will always posses a relatively easily accessible first solution, $y_1(x)$, in the form of modified Taylor series.

$$y_1(x) = \sum_{n=0}^{\infty} a_n \, (x - x_0)^{(n+\nu_0)} \;. \qquad (11.186)$$

The second solution, $y_2(x)$, in contrast, may require more effort.

Comment Regarding Procedure for Working out First Solution

Follow the usual procedure for working out the first solution for (11.185) which is Bessel's equation of order zero. To that end note, the variables $\beta = 0, \gamma = 1, \mu = 1, \nu = 0, \rho = 0$ and the fact that the relevant indicial equation has a double root $\nu_0 = 0$.

Next, use (11.85) that relates a_1 to a_0. That is

$$a_1 = -a_0 \left(\frac{\beta \nu_0 + \nu}{2\nu_0 + \gamma}\right) \;= -a_0 \left(\frac{0+0}{0+1}\right) \;=\; 0 \;. \qquad (11.187)$$

To calculate a_2 and higher-order a_n, for convenience rewrite (11.86) which must obtain for all $n \geq 2$. Even though at the end of these calculations, ν_0 has to be set equal to the double root of the indicial equation—which is $\nu_0 = 0$—for the moment keep the variable ν_0 intact.

$$\begin{aligned} a_n &= -\left[\frac{a_{n-1}\,[\beta\,(n-1+\nu_0)+\nu] + a_{n-2}\;\mu}{(n+\nu_0)(n+\nu_0+\gamma-1)+\rho}\right] \\ &= \;-\left[\frac{a_{n-2}}{(n+\nu_0)^2}\right] \;. \end{aligned} \qquad (11.188)$$

For $n = 2$, (11.188) gives

$$a_2 \;=\; -\left[\frac{a_0}{(2+\nu_0)^2}\right]. \tag{11.189}$$

For $n = 3$, (11.188) and (11.187) lead to

$$a_3 = 0 \tag{11.190}$$

For $n = 4$, (11.188) and (11.189) lead to

$$a_4 \;=\; -\left[\frac{a_2}{(4+\nu_0)^2}\right] \;=\; \left[\frac{a_0}{(4+\nu_0)^2(2+\nu_0)^2}\right] \tag{11.191}$$

and so on. By following the same procedure, one finds for $n = 5, 6, 7, 8, 9,$

$$a_5 \;=\; a_7 \;=\; a_9 \;\ldots\; =\; 0\ , \tag{11.192}$$

$$\begin{aligned} a_6 &= -\left[\frac{a_4}{(6+\nu_0)^2}\right] \;=\; -\left[\frac{a_0}{(6+\nu_0)^2(4+\nu_0)^2(2+\nu_0)^2}\right], \\ a_8 &= -\left[\frac{a_6}{(8+\nu_0)^2}\right] \;=\; \left[\frac{a_0}{(8+\nu_0)^2(6+\nu_0)^2(4+\nu_0)^2(2+\nu_0)^2}\right], \end{aligned} \tag{11.193}$$

and so on.

11.11.1 General Solution

Listed below, in (11.221), is a compact form of the general solution, $G_{\text{BESS0}}(x, \nu_0)$, of Bessel's equation, (11.185), of order zero. This general solution is made up of (11.187) and (11.189)–(11.193), etc.

$$\begin{aligned} &\left(\frac{G_{BESS0}(x,\nu_0)}{a_0\, x^{\nu_0}}\right) \\ &= 1 - \left[\frac{1}{(2+\nu_0)^2}\right]x^2 + \left[\frac{1}{(4+\nu_0)^2(2+\nu_0)^2}\right]x^4 \\ &- \left[\frac{1}{(6+\nu_0)^2(4+\nu_0)^2(2+\nu_0)^2}\right]x^6 \\ &+ \left[\frac{1}{(8+\nu_0)^2(6+\nu_0)^2(4+\nu_0)^2(2+\nu_0)^2}\right]x^8 \end{aligned}$$

$$- \left[\frac{1}{(10+\nu_0)^2(8+\nu_0)^2(6+\nu_0)^2(4+\nu_0)^2(2+\nu_0)^2} \right] x^{10} + O(x^{12}) \tag{11.194}$$

11.11.2 First Solution

The indicial equation for Bessel's equation of order zero has a double root $\nu_0 = 0$. Setting everywhere, in its general solution (11.221), $\nu_0 = 0$ and the arbitrary constant a_0 equal to unity leads to the first solution of Bessel's equation of order zero. Traditionally, it is labeled as $J_0(x)$.

$$\begin{aligned} \frac{G_{BESS0(x,\nu_0=0)}}{a_0} &= J_0(x) \\ &= 1 - \frac{x^2}{(2)^2} + \frac{x^4}{(8)^2} - \frac{x^6}{(48)^2} + \frac{x^8}{(384)^2} - \frac{x^{10}}{(3,840)^2} + O(x^{12}) \\ &= 1 - \frac{1}{(1!^1)} \left(\frac{x}{2}\right)^2 + \frac{1}{(2!^2)} \left(\frac{x}{2}\right)^4 - \frac{1}{(3!^2)} \left(\frac{x}{2}\right)^6 + \frac{1}{(4!^2)} \left(\frac{x}{2}\right)^8 \\ &\quad - \frac{1}{(5!^2)} \left(\frac{x}{2}\right)^{10} + O(x^{12}) \end{aligned} \tag{11.195}$$

These results are suggestive of the well-known expression

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{x}{2}\right)^{2n} \quad . \tag{11.196}$$

11.11.3 Piaggio-Like Second Solution

As expressed in (11.144), the second solution—that is $Y_{\text{BESS0;2;piaggio}}(x, \nu_0)$—is the differential, with respect to ν_0, of the general solution—that means $\left[\frac{d\, G_{BESS0}(x,\nu_0)}{d\, \nu_0} \right]$. Accordingly,

$$\begin{aligned} Y_{BESS0;2;piaggio}(x, \nu_0) &\equiv \left[\frac{d\, G_{BESS0}(x, \nu_0)}{d\, \nu_0} \right] \\ &= G_{BESS0}(x, \nu_0) \cdot \log(x) + \sigma_1\, x^{\nu_0} \left[\frac{2}{(2+\nu_0)^3} x^2 - \frac{4(3+\nu_0)}{(\nu_0^2+6\nu_0+8)^3} x^4 \right] \\ &\quad + \sigma_1\, x^{\nu_0} \left[\frac{\left(88+48\nu_0+6{\nu_0}^2\right)}{(\nu_0^3+12\nu_0^2+44\nu_0+48)^3} \right] x^6 \end{aligned}$$

$$
\begin{aligned}
&- \sigma_1 x^{\nu_0} \left[8 \frac{(100 + 70\nu_0 + 15{\nu_0}^2 + {\nu_0}^3)}{(8+\nu_0)^3(6+\nu_0)^3(4+\nu_0)^3(2+\nu_0)^3} \right] x^8 \\
&- \sigma_1 x^{\nu_0} \left[\frac{2(4,384 + 3,600\nu_0 + 1,020{\nu_0}^2 + 120{\nu_0}^3 + 5{\nu_0}^4)}{(10+\nu_0)^3(8+\nu_0)^3(6+\nu_0)^3(4+\nu_0)^3(2+\nu_0)^3} \right] x^{10} \\
&+ \sigma_1 x^{\nu_0} O(x^{12}) \ldots
\end{aligned} \tag{11.197}
$$

Equation (11.197) contains all the essentials of the second solution. However, in order to convert it into proper form, one needs to do two simple things.

(i) : Replace the arbitrary constants a_0 and σ_1 in (11.197) by unity everywhere.

(ii) : Knowing that the double root of the relevant indicial equation is $\nu_0 = 0$, replace the symbol ν_0 with its value 0 everywhere in (11.197).

Once these changes have been made the second solution, $Y_{\text{BESS0;2;piaggio}}(x, \nu_0 = 0)$, of Bessel's equation of order zero can be written as

$$
\begin{aligned}
&Y_{BESS0;2;piaggio}(x, \nu_0 = 0) \\
&= J_0(x)\log(x) + \frac{x^2}{4} - \left(\frac{3}{128}\right)x^4 + \left(\frac{11}{13,824}\right)x^6 - \left(\frac{25}{1,769,472}\right)x^8 \\
&+ \left(\frac{137}{884,736,000}\right)x^{10} + O(x^{12}) \ .
\end{aligned} \tag{11.198}
$$

Various terms in (11.198) can also be expressed in a form familiar in the literature. For instance, one can write

$$
\begin{aligned}
\frac{x^2}{4} &= (-1)^2 \frac{1}{2^2(1!)^2} x^2 \\
-\left(\frac{3}{128}\right)x^4 &= (-1)^3 \frac{1}{2^4(2!)^2} \left(1 + \frac{1}{2}\right) x^4 \\
\left(\frac{11}{13,824}\right)x^6 &= (-1)^4 \frac{1}{2^6(3!)^2} \left(1 + \frac{1}{2} + \frac{1}{3}\right) x^6 \\
-\left(\frac{25}{1,769,472}\right)x^8 &= (-1)^5 \frac{1}{2^8(4!)^2} \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) x^8 \\
\left(\frac{137}{884,736,000}\right)x^{10} &= (-1)^6 \frac{1}{2^{10}(5!)^2} \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}\right) x^{10}
\end{aligned} \tag{11.199}
$$

The results of the Piaggio-like second solution of Bessel's equation of order zero, given in (11.199), agree with those in the more elegant (11.200) below.

$$Y_{BESS0;2;piaggio}(x, \nu_0 = 0)$$
$$= J_0(x)\log(x) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n!)^2} \left(\frac{x}{2}\right)^{2n} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right). \tag{11.200}$$

As is well known, any linear combination of the first solution—such as (11.196)—and the second solution—such as (11.200)—of a linear ordinary differential equation is also a solution of the same differential equation. Accordingly, the following linear combination of the two is also a second solution. It is traditionally dubbed as $Y_0(x)$, as the second solution of Bessel's equation of order zero.

$$Y_0(x)$$
$$= \frac{2}{\pi} \left\{ J_0(x)\log(x) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n!)^2} \left(\frac{x}{2}\right)^{2n} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) \right\}$$
$$+ \frac{2}{\pi} \{[\gamma_e - \log(2)] J_0(x)\} \quad . \tag{11.201}$$

Here $\gamma_e \approx 0.577215665$ is the so-called Euler's constant.

11.11.4 Solution by Method of Order Reduction

Given the availability of a nonzero solution, $y_1(x)$, of a second-order homogeneous linear ordinary differential equation a linearly independent second solution can be determined by a procedure known as order reduction. [Compare (7.31) to (7.45).] This procedure consists in representing the second solution, $y_2(x)$, as a product of the first solution, which currently is $y_1(x) = J_0(x)$, and an unknown new function $f(x)$.

$$y_2(x) = f(x)\, y_1(x) \quad . \tag{11.202}$$

For a differential equation

$$\left[a_2(x)\, D^2 + a_1(x)\, D + a_0(x)\right] u(x) \;=\; 0 \tag{11.203}$$

the second solution, $y_2(x)$, is given by the relationship

$$y_2(x) = \; y_1(x)\; f(x) \;=\; \sigma_0\; y_1(x) \cdot \int \frac{\exp\left[-\int \frac{a_1(x)}{a_2(x)} \mathrm{d}x\right]}{[y_1(x)]^2}\, \mathrm{d}x \; . \tag{11.204}$$

Note, σ_0 is an arbitrary constant. By comparing (11.203) and the current differential equation (11.185), written below as (11.205),

$$y''(x) + y'(x)\left(\frac{1}{x}\right) + y(x) \;=\; 0 \;, \tag{11.205}$$

we get

$$a_2(x) \;=\; 1 \;;\; a_1(x) \;=\; \left(\frac{1}{x}\right) \;;\; a_0(x) \;=\; 1 \;. \tag{11.206}$$

The inside integral in (11.204) needs to be done first.

$$\exp\left[-\int \frac{a_1(x)}{a_2(x)}\mathrm{d}x\right] \;=\; \exp\left[-\int \left(\frac{1}{x}\right)\mathrm{d}x\right] \;=\; \exp\left[-\log(x)\right] \;=\; \left(\frac{1}{x}\right). \tag{11.207}$$

Use (11.207); insert $y_1(x) = J_0(x)$ in (11.204); and write the resultant as

$$y_2(x) = \sigma_0\, J_0(x)\left[\int \frac{\mathrm{d}x}{x\,\{J_0(x)\}^2}\right] \;. \tag{11.208}$$

Power expand the integrand in (11.208) while using the expansion for $J_0(x)$ given in (11.196),

$$\begin{aligned}\frac{1}{x\,\{J_0(x)\}^2} = &\left(\frac{1}{x}\right) + \left(\frac{1}{2}\right)x + \left(\frac{5}{32}\right)x^3 + \left(\frac{23}{576}\right)x^5 + \left(\frac{677}{73,728}\right)x^7 \\ &+ \left(\frac{7,313}{3,686,400}\right)x^9 + \left(\frac{218,491}{530,841,600}\right)x^{11} + O(x^{13}) \;,\end{aligned} \tag{11.209}$$

and integrate (11.209).

$$\begin{aligned}\int \frac{\mathrm{d}x}{x\,\{J_0(x)\}^2} = \;& \log(x) + \left(\frac{x^2}{4}\right) + \left(\frac{5}{128}\right)x^4 + \left(\frac{23}{3,456}\right)x^6 + \left(\frac{677}{589,824}\right)x^8 \\ &+ \left(\frac{7,317}{36,864,000}\right)x^{10} + \left(\frac{218,491}{6,370,099,200}\right)x^{12} + O(x^{14}) \;.\end{aligned} \tag{11.210}$$

Therefore,

$$\begin{aligned}\frac{y_2(x)}{\sigma_0} \;&=\; J_0(x)\left[\int \frac{\mathrm{d}x}{x\,\{J_0(x)\}^2}\right] \\ &= J_0(x)\log(x) + \left\{1 - \left(\frac{x^2}{4}\right) + \left(\frac{x^4}{64}\right) - \left(\frac{x^6}{2304}\right) + \left(\frac{x^8}{147,456}\right)\right\} \times \\ &\left\{\left(\frac{x^2}{4}\right) + \left(\frac{5}{128}\right)x^4 + \left(\frac{23}{3,456}\right)x^6 + \left(\frac{677}{589,824}\right)x^8 + \left(\frac{7,317}{36,864,000}\right)x^{10}\right\}\end{aligned}$$

$$
\begin{aligned}
&= J_0(x)\log(x) \\
&+ \left(\frac{x^2}{4}\right) - \left(\frac{3}{128}\right)x^4 + \left(\frac{11}{13,824}\right)x^6 - \left(\frac{25}{1,769,472}\right)x^8 \\
&+ \left(\frac{137}{884,736,000}\right)x^{10} \ . \qquad (11.211)
\end{aligned}
$$

Because (11.211) agrees with Piaggio-like second solution of Bessel's equation of order zero—recorded in (11.198)—their final solution is the same. [See *Ross*[17]. Note that Ross' results for $Y_0(x)$ are identical to those derived by using Piaggio-like procedure—refer to (11.201).] That is

$$
\begin{aligned}
&Y_{BESS0;2;ord.reduc}(x, \nu_0 = 0) \\
&= J_0(x)\log(x) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n!)^2}\left(\frac{x}{2}\right)^{2n}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right). \qquad (11.212)
\end{aligned}
$$

11.11.5 Complete Solution

First solution, $J_0(x)$—given in (11.196)—and the second solution, $Y_0(x)$—given in (11.201)—are added together to make complete solution, $Y_{\text{Bess0comp}}$, of Bessel's equation of order zero.

$$
Y_{Bess0comp} \ = \ \sigma_5 J_0(x) \ + \ \sigma_6 Y_0(x) \qquad (11.213)
$$

where σ_5 and σ_6 are arbitrary constants.

11.12 Bessel's Equation of Order n_b

Again consider the differential equation

$$
y''(x) + \left(\frac{\beta x + \gamma}{x}\right) y'(x) + \left(\frac{\mu x^2 + \nu x + \rho}{x^2}\right) y(x) \ = \ 0 \ . \qquad (11.214)
$$

A subset of (11.214) is Bessel's equation of order n_b obtained by setting $\beta = 0, \gamma = 1, \mu = 1, \nu = 0$, and $\rho = -n_b^2$.

$$
y''(x) + \left(\frac{1}{x}\right) y'(x) + \left(\frac{x^2 - n_b^2}{x^2}\right) y(x) \ = \ 0 \, . \qquad (11.215)
$$

Clearly, $x = 0$ is the only singular point for (11.214) and (11.215). And it is a regular singular point. Recall that a regular singular point at $x = 0$ would require $x \times \frac{1}{x}$ and $x^2 \times \frac{1}{x^2}$ both to be analytic at $x = 0$: which is clearly the case here. All the other points are ordinary points.

11.12.1 Indicial Equation

The indicial equation for Bessel's Equation (11.215) of order n_b is

$$\nu_0^2 + \nu_0(\gamma - 1) + \rho \;=\; 0 \;=\; \nu_0^2 + \nu_0(1 - 1) - n_b^2 \;. \tag{11.216}$$

Its roots are:

$$\nu_0 \equiv \;\; \nu_1 \;\; = \;\; n_b \;\; ; \;\; \nu_0 \equiv \;\; \nu_2 \;\; = \;\; -\, n_b \;\; . \tag{11.217}$$

Because one root, $-n_b$, is the negative of the other, n_b, either of the two—say n_b—can be assumed to be positive. Depending on the size of n_b, Bessel's equation of order n_b may belong to any one of the three categories: (1), (2), or (3). Description of what these categories refer to is available contiguous to (11.80)–(11.81), (11.82), and (11.83).

11.12.2 General Solution

For Bessel's equation of order n_b, $\beta = 0, \gamma = 1, \mu = 1, \nu = 0$, and $\rho = -n_b^2$. The relevant differential equations are (11.214) and (11.215). For the general solution, one uses ν_0 as a variable. Its numerical value is specified at the end of the process. Until then proceed as follows.

Use (11.85) that relates a_1 to a_0. That is

$$a_1 = -a_0 \left(\frac{\beta\, \nu_0 + \nu}{2\, \nu_0 + \gamma} \right) \;= -a_0 \left(\frac{0 + 0}{2\nu_0 + 1} \right) \;=\; 0 \;. \tag{11.218}$$

Just like a_1 for odd values of n, $a_n = 0$.

To calculate a_2 and higher-order a_n, it is helpful to rewrite (11.86) that obtains for all even values of n that are ≥ 2.

$$\begin{aligned} a_n &= -\left\{ \frac{a_{n-1}\,[\beta\,(n - 1 + \nu_0) + \nu] + a_{n-2}\;\mu}{(n + \nu_0)(n + \nu_0 + \gamma - 1) + \rho} \right\} \\ &= \;\; -\left\{ \frac{a_{n-2}}{\left[(n + \nu_0)^2 - n_b^2\right]} \right\} \;\; . \end{aligned} \tag{11.219}$$

For even values of n, (11.219) provides the recurrence relationship. And for $n = 2, 4, 6, 8$, etc., it gives

$$\begin{aligned}
\frac{a_2}{a_0} &= -\left\{\left[(2+\nu_0)^2 - n_b^2\right]\right\}^{-1} , \\
\frac{a_4}{a_0} &= \left\{\left[(2+\nu_0)^2 - n_b^2\right]\left[(4+\nu_0)^2 - n_b^2\right]\right\}^{-1} \\
\frac{a_6}{a_0} &= -\left\{\left[(2+\nu_0)^2 - n_b^2\right]\left[(4+\nu_0)^2 - n_b^2\right]\left[(6+\nu_0)^2 - n_b^2\right]\right\}^{-1} \\
\frac{a_8}{a_0} &= \left\{\left[(2+\nu_0)^2 - n_b^2\right]\left[(4+\nu_0)^2 - n_b^2\right]\left[(6+\nu_0)^2 - n_b^2\right]\left[(8+\nu_0)^2 - n_b^2\right]\right\}^{-1} \\
\frac{a_{10}}{a_0} &= -\text{etc.}
\end{aligned} \tag{11.220}$$

General solution, $G_{\text{BESSn}_b}(x, \nu_0)$, of Bessel's equation of order n_b, is made up of (11.218) and (11.220), etc.

$$\begin{aligned}
&\left(\frac{G_{BESSn_b}(x,\nu_0)}{a_0\, x^{\nu_0}}\right) \\
&= 1 - \left\{\frac{1}{\left[(2+\nu_0)^2 - n_b^2\right]}\right\} x^2 + \left\{\frac{1}{\left[(2+\nu_0)^2 - n_b^2\right]\left[(4+\nu_0)^2 - n_b^2\right]}\right\} x^4 \\
&- \left\{\frac{1}{\left[(2+\nu_0)^2 - n_b^2\right]\left[(4+\nu_0)^2 - n_b^2\right]\left[(6+\nu_0)^2 - n_b^2\right]}\right\} x^6 \\
&+ \left\{\frac{1}{\left[(2+\nu_0)^2 - n_b^2\right]\left[(4+\nu_0)^2 - n_b^2\right]\left[(6+\nu_0)^2 - n_b^2\right]\left[(8+\nu_0)^2 - n_b^2\right]}\right\} x^8 \\
&+ O(x^{10})
\end{aligned} \tag{11.221}$$

11.12.3 Two Solutions

Consider a Bessel's equation of order n_b that belongs to category (1): meaning the difference, $2\,n_b$, between the two roots—both of which are real—of its indicial equation is neither zero nor is it equal to an integer. As such, it has two linearly independent solutions, $B_{n_b}(x, \nu_0)$ which are readily found by setting $\nu_0 = \pm n_b$.

11.12.4 First Solution

For the case when the indicial equation roots difference $2n_b$ is not an integer, the first solution, $B_{n_b}(x, \nu_0 = n_b)$, of Bessel's equation of order n_b, is readily found, by

setting $\nu_0 = n_b$ in the general solution $G_{\mathrm{BESSn_b}}(x, \nu_0)$ given in (11.221).

$$\frac{G_{BESSn_b}(x, \nu_0 = n_b)}{a_0\, x^{n_b}} = \frac{B_{n_b}(x, \nu_0 = n_b)}{a_0\, x^{n_b}} = 1 - \frac{x^2}{\left[2^2(1+n_b)\right]} + \frac{x^4}{\left[2^4.2!(2+n_b)(1+n_b)\right]} - \frac{x^6}{\left[2^6.3!(3+n_b)(2+n_b)(1+n_b)\right]} + \frac{x^8}{\left[2^8.4!(4+n_b)(3+n_b)(2+n_b)(1+n_b)\right]} + O(x^{10}) \tag{11.222}$$

Equation (11.222) is suggestive of the more general expression

$$\frac{B_{n_b}(x, \nu_0 = n_b)}{a_0\, x^{n_b}} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^{2n} n!(n+n_b)(n-1+n_b)...(3+n_b)(2+n_b)(1+n_b)}. \tag{11.223}$$

It is tempting to express the denominator of (11.223) by setting

$$2^{2n} n!(n+n_b)(n-1+n_b)...(3+n_b)(2+n_b)(1+n_b) \; . \equiv \frac{2^{2n} n!(n+n_b)!}{n_b!} \tag{11.224}$$

But, alas!, n_b is not an integer and its factorial is not defined. Therefore, as currently written, the relationship (11.224) would be incorrect.

We need to generalize the factorial function to non-integral values of the argument. This can be done through the use of the gamma function which is defined whenever the following integrals converge.

$$\Gamma(z) \;=\; \int_0^{\infty} t^{z-1}\, \exp(-t)\, dt \;=\; \int_0^{1} (-\log t)^{z-1}\, dt \;\; . \tag{11.225}$$

Indeed, $\Gamma(z)$ is valid for all complex values of z: Be they non-integral, negative or imaginary.
The simplest property of the gamma function is the recurrence relationship.

$$\Gamma(z+1) \;=\; z\, \Gamma(z). \tag{11.226}$$

And for integral z

$$\Gamma(z+1) \;=\; z! \tag{11.227}$$

As such, (11.223) may be expressed as

$$B_{n_b}(x, \nu_0 = n_b) = a_0\, 2^{n_b} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n_b + 1)}{n!\,\Gamma(n + n_b + 1)} \left(\frac{x}{2}\right)^{2n+n_b} . \quad (11.228)$$

Part of the right-hand side of (11.228) is function $J_{n_b}(x)$ known as the 'Bessel function of the first kind of order n_b.'

$$J_{n_b}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n + n_b + 1)} \left(\frac{x}{2}\right)^{2n+n_b} . \quad (11.229)$$

11.12.5 Second Solution

The second solution $B_{n_b}(x, \nu_0 = -n_b)$ that refers to $\nu_0 = -n_b$ is now straightforward to determine. For even values of n, the recurrence relationship is

$$\begin{aligned} a_n &= -\left\{ \frac{a_{n-2}}{\left[(n - n_b)^2 - n_b^2\right]} \right\} \\ &= -\frac{a_{n-2}}{n(n - 2\,n_b)} . \end{aligned} \quad (11.230)$$

Hence, the counterparts of (11.222) and (11.223) are

$$\begin{aligned} \frac{B_{n_b}(x, \nu_0 = -n_b)}{a_0\, x^{-n_b}} &= 1 - \frac{x^2}{\left[2^2(1 - n_b)\right]} + \frac{x^4}{\left[2^4.2!(2 - n_b)(1 - n_b)\right]} \\ &- \frac{x6}{\left[2^6.3!(3 + n_b)(2 + n_b)(1 + n_b)\right]} \\ &+ \frac{x^8}{\left[2^8.4!(4 - n_b)(3 - n_b)(2 - n_b)(1 - n_b)\right]} \\ &+ O(x^{10}) \end{aligned} \quad (11.231)$$

Equation (11.231) is suggestive of the more general expression

$$\frac{B_{n_b}(x, \nu_0 = -n_b)}{a_0\, x^{-n_b}} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2\,n}}{2^{2\,n} n!(n - n_b)(n - 1 - n_b)...(3 - n_b)(2 - n_b)(1 - n_b)} . \quad (11.232)$$

It is important to note that because n_b in (11.232) is never a whole number, the denominator in (11.232) does not go to zero. As such there are no pesky divergences in this equation.

Using the same argument that led from (11.222) to (11.228), we can express (11.232) by the following.

$$B_{n_b}(x, \nu_0 = -n_b) \;=\; a_0\, 2^{-n_b} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(-n_b+1)}{n!\,\Gamma(n-n_b+1)} \left(\frac{x}{2}\right)^{2n-n_b} \tag{11.233}$$

Part of the right-hand side of (11.228) is function $J_{n_b}(x)$ known as the 'Bessel function of the first kind of order $-n_b$.'

$$J_{-n_b}(x) \;=\; \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n-n_b+1)} \left(\frac{x}{2}\right)^{2n-n_b} . \tag{11.234}$$

11.12.6 Complete Solution

Complete solution of differential equation (11.215) for the case where the indicial equation roots differ by non-integer—meaning the Bessel's equation is of category (1)—is the sum of the first- and the second solution that were given in (11.228) and (11.233).

$$B_{n_b}(x, \nu_0 = n_b) \;+\; B_{n_b}(x, \nu_0 = -n_b) = \; \sigma_1 \left(\frac{x}{2}\right)^{2n+n_b} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n_b+1)}{n!\,\Gamma(n+n_b+1)} 2^{n_b}$$

$$+\, \sigma_2 2^{-n_b} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(-n_b+1)}{n!\,\Gamma(n-n_b+1)} \left(\frac{x}{2}\right)^{2n-n_b} . \tag{11.235}$$

11.13 Bessel's Indicial Equation

Roots are Equal

If the two roots, n_b and $-n_b$, of the indicial equation are equal, they are both zero. As such the Bessel function is of order zero and has already been analyzed in detail in (11.185)–(11.213).

Roots Differ by Integer

Given that n_b is real and positive and Bessel's indicial equation roots, n_b and $-n_b$, differ by an integer, there are two possibilities. Either n_b is an integer or it is an half-odd integer.

Roots Are Integers

Differential equations whose indicial equation roots are integers have an important feature. Their first solution is readily worked out by following the usual technique. In contrast, their second solution tends to have recurring zeros in the denominator leading to divergences. Simplest example of this happenstance is a Bessel's equation of order unity.

11.13.1 Bessel's Equation of Order Unity

First Solution

Replacing n_b^2 by unity transforms Bessel's equation of order n_b—see (11.215)—into Bessel's equation of order unity—see (11.236).

$$y''(x) + \left(\frac{1}{x}\right) y'(x) + \left(\frac{x^2 - 1}{x^2}\right) y(x) \;=\; 0 \;. \tag{11.236}$$

The two roots of the indicial equation of (11.236) are $n_b = 1$ and $n_b = -1$. Its first solution must use the larger root, namely $n_b = 1$. Because the general solution $G_{\mathrm{BESSn_b}}(x, \nu_0)$—see (11.221)—is still applicable, as are other equations leading to (11.228), all that is needed to convert (11.228) into the first solution, $B_{1;x}(n_b = 1)$, of (11.236) is change the variable n_b to unity. This leads to the result

$$\begin{aligned} B_{1;x}(n_b = 1) \;&=\; a_0\, 2^1 \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(1+1)}{n!\,\Gamma(n+1+1)} \left(\frac{x}{2}\right)^{2n+1} \\ = 2\, a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(n+1)!} \left(\frac{x}{2}\right)^{2n+1} \;&\equiv\; 2\, a_0\, J_1(x)\;, \end{aligned} \tag{11.237}$$

where $J_1(x)$ is Bessel's function of the first kind of order unity. [Compare (11.229).]

Second Solution

Bessel's equation of order unity is treated in *Piaggio*[10], pp. 114–115. Because the Piaggio book is not readily available, and the presentation there is minimal and somewhat abstruse, it is helpful to record a detailed solution here.

Consider the general solution, (11.221), of Bessel's equation of order n_b and convert it to general solution of Bessel's equation of order unity by setting $n_b = 1$.

$$\begin{aligned} & G_{BESS1}(x, \nu_0) \\ &= a_0 x^{\nu_0} [1 - \left\{ \frac{x^2}{(\nu_0+3)(\nu_0+1)} \right\} + \left\{ \frac{x^4}{(\nu_0+5)(\nu_0+3)^2(\nu_0+1)} \right\} \\ &- \left\{ \frac{x^6}{(\nu_0+7)(\nu_0+5)^2(\nu_0+3)^2(\nu_0+1)} \right\} \\ &+ \left\{ \frac{x^8}{(\nu_0+9)(\nu_0+7)^2(\nu_0+5)^2(\nu_0+3)^2(\nu_0+1)} \right\} \\ &+ O(x^{10})] \end{aligned} \tag{11.238}$$

When $\nu_0 = 1$, (11.238) is the first solution, $B_{1;x}(n_b = 1)$, of Bessel's equation of order unity. [Reminder: The first solution, $B_{1;x}(n_b = 1)$, of Bessel's equation of order unity is (11.237).] On the other hand, when ν_0 is set equal to -1, the denominator of the general solution (11.238) goes to zero because of the presence of the factor $(\nu_0 + 1)$ everywhere. To avoid this difficulty, Piaggio proposed to change the arbitrary constant a_0 to another arbitrary constant $a_0\,(\nu_0 + 1)$. The *Piaggio*[10.] version of the general solution is

$$\begin{aligned} & G_{BESSpiaggio}(x, \nu_0) \\ &= a_0 x^{\nu_0} [(\nu_0 + 1) - \left\{ \frac{x^2}{(\nu_0+3)} \right\} \\ &+ \left\{ \frac{x^4}{(\nu_0+5)(\nu_0+3)^2} \right\} \\ &- \left\{ \frac{x^6}{(\nu_0+7)(\nu_0+5)^2(\nu_0+3)^2} \right\} \\ &+ \left\{ \frac{x^8}{(\nu_0+9)(\nu_0+7)^2(\nu_0+5)^2(\nu_0+3)^2} \right\} \\ &+ O(x^{10})] \end{aligned} \tag{11.239}$$

The important part of Piaggio's suggestion is the following: the second solution, $B_{2;x}(n_b)$, of Bessel's equation of order unity is the differential with respect to the variable ν_0 of its general solution $G_{\text{BESSpiaggio}}(x, \nu_0)$. The final requirement is that the result of such differential be evaluated at $\nu_0 = -1$ which is the second root of the relevant indicial equation. Recall (11.121), whereby

$$\left(\frac{\mathrm{d}x^{\nu_0}}{d\nu_0} \right) = x^{\nu_0} \log(x) \ , \tag{11.240}$$

and proceed as follows:

$$\begin{aligned}\frac{dG_{BESSpiaggio}(x,\nu_0)}{d\nu_0} &= \log(x)\, G_{BESSpiaggio}(x,\nu_0)\\ &+ a_0\, x^{\nu_0}[1+\left\{\frac{1}{(\nu_0+3)^2}\right\}x^2-\left\{\frac{13+3\nu_0}{(\nu_0+5)^2(\nu_0+3)^3}\right\}x^4\\ &-\left\{\frac{127+52\nu_0+5\nu_0^2}{(\nu_0+7)^2(\nu_0+5)^3(\nu_0+3)^3}\right\}x^6\\ &-\left\{\frac{1383+753\nu_0+129\nu_0^2+7\nu_0^3}{(\nu_0+9)^2(\nu_0+7)^3(\nu_0+5)^3(\nu_0+3)^3}\right\}x^8\\ &+O(x^{10})\end{aligned} \tag{11.241}$$

Setting $\nu_0 = -1$ leads to the second solution $B_{2;x}(n_b = -1)$.

$$\begin{aligned}B_{2;x}(n_b=-1) &= \log(x)\, B_{1;x}(n_b=1)\\ &+\left(\frac{a_0}{x}\right)\left[1+\left(\frac{x^2}{4}\right)-\left(\frac{5}{64}\right)x^4+\left(\frac{5}{1152}\right)x^6-\left(\frac{47}{442,368}\right)x^8+O(x^{10}).\right]\end{aligned} \tag{11.242}$$

Chapter 12
Answer to Assigned Problems



12.1 Problems Group I, 3-chapt

[Note: Look just below (3.42).]
S_{comp} for problems (1)–(10)

$$\sigma_0 \exp(x) + \sigma_1 \exp(-3x) \tag{1}$$

$$\sigma_0 \exp(4x) + \sigma_1 \exp(-x) \tag{2}$$

$$(\sigma_0 + \sigma_1 x) \exp\left(-\frac{x}{2}\right) \tag{3}$$

$$(\sigma_0 + \sigma_1 x) \exp\left(\frac{x}{2}\right) \tag{4}$$

$$(\sigma_0 + \sigma_1 x + \sigma_2 x^2) \exp(-5x) \tag{5}$$

$$(\sigma_0 + \sigma_1 x + \sigma_2 x^2) \exp(2x) \tag{6}$$

$$\exp\left(-\frac{x}{2}\right)\left[\sigma_1 \sin\left(\frac{\sqrt{3}}{2}x\right) + \sigma_2 \cos\left(\frac{\sqrt{3}}{2}x\right)\right] \tag{7}$$

$$\exp\left(\frac{x}{2}\right)\left[\sigma_1 \sin\left(\frac{\sqrt{3}}{2}x\right) + \sigma_2 \cos\left(\frac{\sqrt{3}}{2}x\right)\right] \tag{8}$$

$$\exp(-x)\left[\sigma_1 \sin\left(\sqrt{2}\,x\right) + \sigma_2 \cos\left(\sqrt{2}\,x\right)\right] \tag{9}$$

$$\exp\left(-\frac{3}{2}x\right)\left[\sigma_1 \sin\left(\frac{\sqrt{7}}{2}x\right) + \sigma_2 \cos\left(\frac{\sqrt{7}}{2}x\right)\right] \tag{10}$$

12.2 Problems Group II, 3-chapt

[Note: Look above (3.74).]

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I_{pi} for problems (1)–(10)

(1) $-\frac{c_0}{3} - \frac{c_1}{27}\left[40 + 42x + 18x^2 + 9x^3\right]$

(2) $-\frac{c_0}{4} + \frac{c_2}{128}\left[153 - 156x + 72x^2 - 32x^3\right]$

(3) $4c_0 + 4c_3\left[-192 + 72x - 12x^2 + x^3\right]$

(4) $4c_0 + 4c_4\left[1920 + 768x + 144x^2 + 16x^3 + x^4\right]$

(5) $\frac{c_0}{8} + \frac{c_5}{16}\left[45 - 60x + 36x^2 - 12x^3 + 2x^4\right]$

(6) $-\frac{c_0}{8} - \frac{c_6}{16}\left[45 + 60x + 36x^2 + 12x^3 + 2x^4\right]$

(7) $c_0 + c_7\left[-120x + 60x^2 - 5x^4 + x^5\right]$

(8) $c_0 + c_8\left[-120x - 60x^2 + 5x^4 + x^5\right]$

(9) $\frac{81c_0}{243} + \frac{c_9}{243}\left[400 - 1320x + 720x^2 + 180x^3 - 270x^4 + 81x^5\right]$

(10) $\frac{256c_0}{1024} + \frac{c_{10}}{1024}\left[-4095 = 8100x - 3960x^2 - 1440x^3\right]$
$+\frac{c_{10}}{1024}\left[2400x^4 - 1152x^5 + 256x^6\right]$

12.3 Problems Group III, 3-chapt

[Note: Look just below (3.80).]

Particular integrals, I_{pi}, for problems (1)–(10) are given below.

$$-\frac{c_0}{3} + \frac{c_1 \exp(\alpha t)}{\alpha^2 + 2\alpha - 3} \quad . \tag{1}$$

$$-\frac{c_0}{4} + \frac{c_2 \exp(\alpha t)}{\alpha^2 - 3\alpha - 4} \quad . \tag{2}$$

$$4c_0 + \frac{4c_3 \exp(\alpha t)}{4\alpha^2 + 4\alpha + 1} \quad . \tag{3}$$

$$4c_0 + \frac{4c_4 \exp(\alpha t)}{4\alpha^2 - 4\alpha + 1} \quad . \tag{4}$$

$$\frac{c_0}{8} + \frac{c_5 \exp(\alpha t)}{\alpha^3 + 6\alpha^2 + 12\alpha + 8} \quad . \tag{5}$$

$$-\frac{c_0}{8} + \frac{c_6 \exp(\alpha t)}{\alpha^3 - 6\alpha^2 + 12\alpha - 8} \quad . \tag{6}$$

$$c_0 + \frac{c_7 \exp(\alpha t)}{\alpha^2 + \alpha + 1} \quad . \tag{7}$$

$$c_0 + \frac{c_8 \exp(\alpha t)}{\alpha^2 - \alpha + 1} \quad . \tag{8}$$

$$\frac{c_0}{3} + \frac{c_9 \exp(\alpha t)}{\alpha^2 + 2\alpha + 3} \quad . \tag{9}$$

$$\frac{c_0}{4} + \frac{c_{10} \exp(\alpha t)}{\alpha^2 + 3\alpha + 4} \quad . \tag{10}$$

12.4 Problems Group IV, 3-chapt

[Note: Look just below (3.83).]

I_{pi} for problems (1)–(10)

$$\frac{\sin(x) - 2\cos(x)}{5} \quad . \tag{1}$$

$$\left(\frac{-i}{17}\right)[5\sin(x) - 3\cos(x)] \quad . \tag{2}$$

$$\frac{8}{25}\sin(x) - \frac{24}{1369}\sin(3x) - \frac{6}{25}\cos(x) + \frac{70}{1369}\cos(3x) \quad . \tag{3}$$

$$\frac{6}{25}\sin(x) - \frac{70}{1369}\sin(3x) - \frac{8}{25}\cos(x) + \frac{24}{1369}\cos(3x) \quad . \tag{4}$$

$$\frac{1}{4} + \frac{\sin(2\,x) - \cos(2\,x)}{16} \quad . \tag{5}$$

$$\frac{1}{4} + \frac{\sin(2\,x) + \cos(2\,x)}{16} \quad . \tag{6}$$

$$\frac{4\sin(2x) - 6\cos(2x)}{13} \quad . \tag{7}$$

$$-\left(\frac{2\,i}{13}\right)[3\sin(2x) - 2\cos(2x)] \quad . \tag{8}$$

$$\frac{2}{3} + \frac{16\sin(4x) - 26\cos(4x)}{233} \quad . \tag{9}$$

$$-\frac{1}{2} + \frac{\sin(4x) - \cos(4x)}{12} \quad . \tag{10}$$

12.5 Problems Group V, 3-chapt

[Note: Look just below (3.91).]

I_{pi} for problems (1)–(10)

$$\frac{\exp(2x)}{169}\{[-4i+39x]\sin(x)+[3+26x]\cos(x)\}\ . \quad (1)$$

$$\left(\frac{-i\ \exp(2x)}{125}\right)\{[2+35x]\sin(x)+[11+5x]\cos(x)\}\ . \quad (2)$$

$$\frac{2}{25}[3\sin(x)+4\cos(x)]-\frac{2}{1369}[35\sin(3x)+12\cos(3x)]\ . \quad (3)$$

$$-\frac{2}{25}[4\sin(x)+3\cos(x)]+\frac{2}{1369}[12\sin(3x)+35\cos(3x)]\ . \quad (4)$$

$$-\left(\frac{\exp(x}{625}\right)\{72\sin(x)+21\cos(x)-x[65\sin(x)+45\cos(x)]\}\ . \quad (5)$$

$$\exp(x)\{3\sin(x)+x\,[\sin(x)-\cos(x)]\}\ . \quad (6)$$

$$\frac{2}{3}\exp(x)(1-x)-\frac{2\ \exp(x)}{1369}[(104-222x)\sin(2x)-(153-37x)\cos(2x)]\ . \quad (7)$$

$$-2\exp(x)(1-x)+\frac{2\ \exp(x)}{169}[(32+26x)\sin(2x)+(43-39x)\cos(2x)]\ . \quad (8)$$

$$4\ \exp(-x)[2\sin(x)+x\ \cos(x)]\ . \quad (9)$$

$$\frac{2\exp(x)}{1369}[(130-259x)\sin(x)-(151-185x)\cos(x)]\ . \quad (10)$$

12.6 Problems Group VI, 3-chapt

[Note: Look at (3.143).]

Given below are the separated simultaneous linear ordinary equations, their S_{comp} and I_{pi} for problems (1)–(3), and some help with the solution of problems (4)–(6).

$$(2\Delta^2+2\Delta+1)x\ =\ 6\exp(t)-3\,t\ \exp(t)\quad ,$$

$$(2\Delta^2+2\Delta+1)y\ =\ 2\exp(t)\,t\ -2\ \exp(t)\quad .$$

$$S_{comp;\,x}\ =\ \exp\left(-\frac{t}{2}\right)\left\{\sigma_1\sin\left(\frac{t}{2}\right)+\sigma_2\cos\left(\frac{t}{2}\right)\right\}\quad ,$$

$$S_{comp;\,y}\ =\ \exp\left(-\frac{t}{2}\right)\left\{\sigma_3\sin\left(\frac{t}{2}\right)+\sigma_4\cos\left(\frac{t}{2}\right)\right\}\quad .$$

$$I_{pi\,;\,x}\ =\ \frac{3}{5}\exp(t)\left\{\frac{16}{5}-t\right\}\ ,\ I_{pi\,;\,y}\ =\ \frac{2}{5}\exp(t)\left\{-\frac{11}{5}+t\right\}\quad ;$$

$$\sigma_3\ =\ -\frac{1}{5}(2\sigma_1-\sigma_2)\ ;\ \sigma_4\ =\ -\frac{1}{5}(\sigma_1+2\sigma_2)\ .\quad (1)$$

$$
\begin{aligned}
(3\Delta^2 - 3)x &= 2\exp(t) + 3t\,\exp(t) - 2t - 1 \quad , \\
(3\Delta^2 - 3)y &= -\exp(t) - 3t\exp(t) + t + 2 \quad . \\
S_{comp;\,x} &= \sigma_1 \exp(-t) + \sigma_2 \exp(t) \quad , \\
S_{comp;\,y} &= \sigma_3 \exp(-t) + \sigma_4 \exp(t) \quad , \\
I_{pi\,;\,x} &= \frac{\exp(t)}{24}\left(-1 + 2t + 6t^2\right) + \frac{1}{3}(1 + 2t) \quad , \\
I_{pi\,;\,y} &= \frac{\exp(t)}{24}\left(-1 + 2t - 6t^2\right) - \frac{1}{3}(2 + t) \quad ; \\
\sigma_3 &= \sigma_1 \; ; \; \sigma_4 = -\sigma_2 \quad . \qquad (2) \\
(\Delta^2 + 2\Delta + 2)x &= 1 + 2t - \exp(t) - 2t\,\exp(t) \quad , \\
(\Delta^2 + 2\Delta + 2)y &= 3\exp(t) + 3t\exp(t) - 1 \quad . \\
S_{comp;\,x} &= \exp(-t)\left[\sigma_1 \sin(t) + \sigma_2 \cos(t)\right] \quad , \\
S_{comp;\,y} &= \exp(-t)\left[\sigma_3 \sin(t) + \sigma_4 \cos(t)\right] \quad , \\
I_{pi\,;\,x} &= \frac{\exp(t)}{25}(3 - 10t) - \frac{1}{2} + t \quad , \\
I_{pi\,;\,y} &= \frac{\exp(t)}{25}(3 + 15t) - \frac{1}{2} \quad ; \\
\sigma_3 &= \sigma_1 \; ; \; \sigma_4 = \sigma_2 \quad . \qquad (3)
\end{aligned}
$$

In problem (4) set $(-x + 3) = \exp(-t)$, $(-x + 3)D = \Delta$, and thereby transform it into problem (1).

In problem (5) set $(2x - 1) = \exp(2t)$, $(2x - 1)D = \Delta$, and thereby transform it into problem (2).

In problem (6) set $(x + 1) = \exp(t)$, $(x + 1)D = \Delta$ and $(x + 1)^2 D^2 = \Delta(\Delta - 1)$ and thereby transform it into problem (3).

12.7 Problems Group I, 4-chapt

[Note: Look at (4.18).]

Solution $u = u(x)$ to problem set I problems (1)–(12) is given below.

$$
\begin{aligned}
u(x) &= \left(\frac{1}{x}\right)\left[\frac{x^3}{3} + \sigma_0\right] . \quad (1) \\
u(x) &= \frac{1}{x^3}\left[\frac{x^6}{6} + \sigma_0\right] . \quad (2) \\
u(x) &= \frac{1}{x^2 \exp(x)}\left[3\exp(x)(x - 1) + \sigma_0\right] . \quad (3)
\end{aligned}
$$

$$u(x) = \frac{1}{x\exp(3x)}\left[\frac{\exp(4x)}{32}(8x^2-4x+1)+\sigma_0\right]. \quad (4)$$

$$u(x) = \left(\frac{1}{\sin x}\right)\left[-\log(\cos x)+\sigma_0\right]. \quad (5)$$

$$u(x) = (\cos x)\left[x+\sigma_0\right]. \quad (6)$$

$$u(x) = \frac{1}{(x-1)\exp(x)}\left[\frac{1}{2}\exp(2x)(x-1)^2+\sigma_0\right]. \quad (7)$$

$$u(x) = \frac{1}{(\sin x)^3}\left[\frac{(\sin x)^4}{2}+\sigma_0\right]. \quad (8)$$

$$u(x) = (x+1)\left[\frac{(x^2+2x)}{2}+\sigma_0\right]. \quad (9)$$

$$u(x) = (x-1)\left[\frac{(x^2-2x)}{2}+\sigma_0\right]. \quad (10)$$

$$u(x) = \left(\frac{1}{\log x}\right)^2\left[\frac{2x^2(\log x)^2-2x^2\log(x)+x^2}{4}+\sigma_0\right]. \quad (11)$$

$$u(x) = \frac{1}{x(\sin x)}\left[x\sin(x)+\cos(x)+\sigma_0\right]. \quad (12)$$

12.8 Problems Group II, 4-chapt

[Note: Look at (4.78).]

Solution $u(x)$ to problem set II, problems (1)–(10) is given below.

$$u(x) = \pm\left[\left(\frac{1}{3}\right)(2+\sigma_0\exp(-6x))\right]^{\frac{1}{2}}. \quad (1)$$

$$u(x) = \frac{2}{3+\sigma_0\exp(x^2)}. \quad (2)$$

$$u(x) = \pm\left[\frac{3}{2+\sigma_0\exp(3x^2)}\right]^{\frac{1}{2}}. \quad (3)$$

$$u(x) = \pm\left[\frac{1}{\exp(x^2)\{-6x+\sigma_0\}}\right]^{\frac{1}{2}}. \quad (4)$$

$$u(x) = \left[\frac{1}{\exp(x^3)\{-18x+\sigma_0\}}\right]^{\frac{1}{3}}; \quad (5)$$

$$u(x) = (-1)^{\frac{2}{3}}\left[\frac{1}{\exp(x^3)\{-18x+\sigma_0\}}\right]^{\frac{1}{3}}; \quad (5)$$

$$u(x) = (-1)^{\frac{4}{3}} \left[\frac{1}{\exp(x^3)\{-18\,x + \sigma_0\}} \right]^{\frac{1}{3}} \quad . \quad (5)$$

$$u(x) = \pm \left[2\,x^2 - 1 + \sigma_0 \exp(-2x^2) \right]^{\frac{1}{4}} \quad ; \quad (6)$$

$$u(x) = \pm\,(i) \left[2\,x^2 - 1 + \sigma_0 \exp(-2x^2) \right]^{\frac{1}{4}} \quad . \quad (6)$$

$$u(x) = \frac{1}{x(-3\,x + \sigma_0)} \quad . \quad (7)$$

$$u(x) = \frac{1}{x^2[-2\,\log(x) + \sigma_0]} \quad . \quad (8)$$

$$u(x) = \pm \left[\frac{1}{2x^2 + \sigma_0\,x^4} \right]^{\frac{1}{2}} \quad . \quad (9)$$

$$u(x) = \frac{\sqrt[3]{9\,x^8 + \sigma_1}}{2\,x^2} \quad ; \quad (10)$$

$$u(x) = (-1)^{\frac{2}{3}} \frac{\sqrt[3]{9\,x^8 + \sigma_1}}{2\,x^2} \quad ; \quad (10)$$

$$u(x) = (-1)^{\frac{4}{3}} \frac{\sqrt[3]{9\,x^8 + \sigma_1}}{2\,x^2} \quad . \quad (10)$$

12.9 Problems Group I, 6-chapt

[Note: Look just below (6.18).]

General Solution

$$y = x\,\sigma + 3\,\sigma^2 \quad . \quad (1)$$

$$y = x\,\sigma + 3\,\sigma^3 \quad . \quad (2)$$

$$y = x\,\sigma - 2\,\sin\sigma \quad . \quad (3)$$

$$y = x\,\sigma + 2\,\cos\sigma \quad . \quad (4)$$

Singular Solution

$$12\,y + x^2 = 0 \quad . \quad (1)$$

$$81\,y^2 + 4\,x^3 = 0 \quad . \quad (2)$$

$$4 - x^2 = \left[y - x\,\cos^{-1}\left(\frac{x}{2}\right) \right]^2 \quad . \quad (3)$$

$$4 - x^2 = \left[y - x\,\sin^{-1}\left(\frac{x}{2}\right) \right]^2 \quad . \quad (4)$$

12.10 Problems Group II, 6-chapt

[Note: Look just below (6.31).]

General Solution

$$x = \sigma_1 \sqrt{q} - 18\,q \ . \ (1)$$

$$y = \sigma_1 \left(\frac{q^{\frac{3}{2}}}{3}\right) - 9\,q^2 \ . \ (1)$$

$$x = \sigma_1\, q^{\frac{1}{4}} - 2\left(\frac{q}{3}\right) \ . \ (2)$$

$$y = \sigma_1 \left(\frac{q^{\frac{5}{4}}}{5}\right) - \left(\frac{q^2}{3}\right) \ . \ (2)$$

$$x = \left(\frac{\sigma_1}{\sqrt{q^3}}\right) - \left(\frac{5}{q}\right) \ . \ (3)$$

$$y = 3\left(\frac{\sigma_1}{\sqrt{q}}\right) + 5\left[\log(q) - 3\right] \ . \ (3)$$

Singular Solution

Singular solution to (1), (2), and (3) is the same. That is:

$$y(x) = 0 \ .$$

12.11 Problems Group III, 6-chapt

[Note: Look just below (6.35).]

$$(1): \quad \int \left[\frac{\mathrm{d}y}{y}\right] = \int \left[\frac{\mathrm{d}x}{x}\right] \ ; \ \log(y) = \log(x) + \text{const} \ .$$

$$Or \ \ Equivalently \ : \ y = \sigma_0\, x \ .$$

$$(2): \quad \int \left[\frac{\mathrm{d}y}{\exp(3y)}\right] = \int \exp(-2x)\,\mathrm{d}x \ ; \ \frac{\exp(-3y)}{-3} = \frac{\exp(-2x)}{-2} + \text{const} \ .$$

$$Or \ \ Equivalently \ : \ y = -\frac{1}{3}\log\left[\frac{3}{2}\exp(-2x) + \sigma_0\right] \ .$$

$$(3): \quad \int \left[\frac{dy}{y}\right] = \int \left[2x\ \exp(x^2)\right] dx\ ; \quad \log(y) = \exp(x^2) + \sigma_0\ .$$

$$Or\ \ Equivalently\ :\ \ y = \log^{-1}[\exp(x^2) + \sigma_0]\ .$$

$$(4): \quad \int \left[\frac{dy}{y^2}\right] = \int \left[x^2\right] dx\ ; \quad \log(y) = \exp(x^2) + \sigma_0\ .$$

$$Or\ \ Equivalently\ :\ \ y = \log^{-1}[\exp(x^2) + \sigma_0]\ .$$

12.12 Problems Group IV, 6-chapt

[Note: Look just below (6.41)]

$$(1): \quad y = -x + \frac{x}{1 - \sigma_0\, x}\ .$$

$$(2): \quad y = x\ \sin^{-1}\left[(\sigma_0\, x)^3\right]\ .$$

$$(3): \quad y = 5\, x\left[\sigma_0 + \log x\right]\ .$$

$$(4): \quad y = \pm\, x\, \sqrt{8}\ \left[\sigma_0 + \log x\right]\ .$$

$$(5): \quad 4\, y = x\ \left[3\ \log(\sigma_0\, x) \pm \sqrt{\{3\ \log(\sigma_0\, x)\}^2 + 8}\right]\ .$$

12.13 Problems Group V, 6-chapt

[Note: Look just below (6.73).]

$$(1): \quad y = (x+1)\ \tan\{\log(x+1) + \sigma_0\}\ .$$

$$(2): \quad (y+2) = (x+1)[\log(x+1) + \sigma_0]\ .$$

$$(3): \quad y = (x + \sigma_0)\ \cos x\ .$$

$$(4): \quad (y+2) = (x+3)\log[\sigma_0 + \log(x+3)]\ .$$

$$(5): \quad 2\, y = -x - 3 \pm \sqrt{3\, x^2 + 10\, x + \sigma_0}\ .$$

$$(6): \quad 3\, y = -x - 4 \pm \sqrt{4\, x^2 + 14\, x + \sigma_0}\ .$$

$$(7): \quad 3\, y = x + \frac{1}{2}\log\left[2(x+y) + 3\right] + \sigma_0\ .$$

12.14 Problems Group VI, 6-chapt

[Note: Look just below (6.91).]

$$(1) \ : \qquad y \ = \ \frac{\sigma_3 - x}{x^2 + 2} \ .$$

$$(2) \ : \qquad \sigma_3 \ = \ x(1 + y + y^2) \ : \ or \ \ equivalently,$$

$$y \ = \ \frac{1}{2}\left[-1 \pm \sqrt{-3 + \frac{4\sigma_3}{x}}\right] \ .$$

$$(3) \ : \qquad \sigma_3 \ = \ \frac{x^2}{2} \ + \ \frac{y^2}{2} \ + \ x \ \exp(y) \ .$$

12.15 Problems Group VII, 6-chapt

[Note: See (6.241).]

Procedure: First choose $u(x, y)$ and $v(x, y)$ from one of the three equations in problem set VI chapter (ST). Then, just to be sure, check, by using (6.82), as to whether the given differential equation is exact or inexact. And if it is inexact, employ either (6.101) or (6.103) to find whether an integrating factor that depends only upon a single variable is possible. When the equation has been made exact, solve it according to (6.88) and (6.91).

$$(1) \ : \qquad y \ = \ \frac{\sigma_0}{(2x + 3)^{\frac{3}{2}}} - \frac{2}{3} \ .$$

$$(2) \ : \qquad y \ = \ \frac{\sigma_0}{\sqrt{x}} \ .$$

$$(3) \ : \qquad y \ = \ \sigma_0 \exp\left(-\frac{3x^2}{2}\right) \ + \ \frac{2}{3} \ - \ x^2 \ .$$

12.16 Problems Group VIII, 6-chapt

[Note: Look just below (6.268).]

$$(1): \quad y = -1 + \frac{1}{x + \sigma_0}$$

$$(2): \quad y = -\frac{1}{2} + \frac{\sqrt{7}}{2} \tan\left(\sigma_0 - \frac{\sqrt{7}}{2}x\right) \ .$$

12.17 Problems Group IX, 6-chapt

[Note: Look just below (6.315).]

Given below are the S_{comp} and I_{pi} for problems (1)–(5) and some help with the solution.

$$Set\ \ (x+3)\ =\ \exp(t)\ ;\ (x+3)\,D\ =\ \Delta\ ;\ (x+3)^2\,D^2\ =\ \Delta(\Delta-1)\ \ .$$

$$Therefore,\ \ (1)\ :\ (\Delta^2+\Delta)u\ =\ -\,t\,\exp(t)\ ,$$

$$S_{comp;\,t}\ =\ \sigma_0+\sigma_1\exp(-t)\ \ ,\ \ I_{pi;\,t}\ =\ \exp(t)\left[\frac{3}{4}-\frac{t}{2}\right]\quad ,$$

$$S_{comp;\,x}\ =\ \sigma_0+\frac{\sigma_1}{x+3}\ \ ,\ \ I_{pi;\,x}\ =\ (x+3)\left[\frac{3}{4}-\frac{\log(x+3)}{2}\right]\quad ,\ (1)$$

$$and\ \ (2)\ :\ (\Delta^2+\Delta+3)u\ =\ -\,t\,\exp(t)\ ,$$

$$S_{comp;\,t}\ =\ \exp\left(-\frac{t}{2}\right)\left[\sigma_1\sin\left(\frac{\sqrt{13}}{2}t\right)+\sigma_2\cos\left(\frac{\sqrt{13}}{2}t\right)\right]\ ,$$

$$I_{pi;\,t}\ =\ \exp(t)\left[\frac{3}{25}-\frac{t}{5}\right]\ ,$$

$$S_{comp;\,x}\ =\ \frac{1}{\sqrt{x+3}}\left[\sigma_1\sin\left(\frac{\sqrt{13}}{2}\log(x+3)\right)+\sigma_2\cos\left(\frac{\sqrt{13}}{2}\log(x+3)\right)\right]$$

$$I_{pi;\,x}\ =\ (x+3)\left[\frac{3}{25}-\frac{\log(x+3)}{5}\right]\ .\qquad (2)$$

$$Set\ \ (2x-1)\ =\ \exp(2t)\ ;\ (2x-1)\,D\ =\ \Delta\ ;\ (2x-1)^2\,D^2\ =\ \Delta(\Delta-1)\ \ .$$

$$Therefore,\ \ (3)\ :\ (\Delta^2-4\Delta)u\ =\ -\,4t\,\exp(2t)\sin(2t)\ ,$$

$$S_{comp;\,t}\ =\ \sigma_0+\sigma_1\exp(4t)\ ,$$

$$I_{pi;\,t}\ =\ \frac{\exp(2t)}{4}\left[\cos(2t)+2t\,\sin(2t)\right]\quad ,$$

$$S_{comp;\,x}\ =\ \sigma_0+\sigma_1(2x-1)^2\ ,$$

$$I_{pi;\,x}\ =\ \frac{(2x-1)}{4}\left\{\cos[\log(2x-1)]+\log(2x+1)\sin[\log(2x-1)]\right\}\quad ,\ (3)$$

$$and\ \ (4)\ :\ (\Delta^2-4\Delta+4)u\ =-4t\,\exp(2t)\sin(2t)\ ,$$

$$S_{comp;\,t}\ =\ \exp(2t)\left[\sigma_1\,t+\sigma_0\right]\ ,$$

$$I_{pi;\,t}\ =\ \exp(2t)\left[\cos(2t)+t\,\sin(2t)\right]\quad ,$$

$$S_{comp;\,x}\ =\ (2x-1)\left[\sigma_2\,\log(2x-1)+\sigma_4\right]\ ,$$

$$I_{pi;\,x}\ =\ \frac{(2x-1)}{2}\left[2\cos(\log(2x-1))+\log(2x-1)\,\sin(\log(2x-1))\right]\ .\qquad (4)$$

$$Set\ \ (3x+2)\ =\ \exp(3t)\ ;\ (3x+2)\,D\ =\ \Delta\ ;\ (3x+2)^2\,D^2\ =\ \Delta(\Delta-3)\ \ .$$

$$Therefore,\ \ (5)\ :\ \left[\Delta^2-6\Delta+\frac{1}{4}\right]u\ =\ \exp(3t)\sin(3t)\ ,$$

$$S_{comp;\,t}\ =\ \exp(3t)\left[\sigma_1\exp\left(\frac{\sqrt{35}}{2}t\right)+\sigma_2\exp\left(-\frac{\sqrt{35}}{2}t\right)\right]\ ,$$

$$
\begin{aligned}
I_{pi;\,t} &= -\frac{4\exp(3t)}{71}\sin(3t) \quad , \\
S_{comp;\,x} &= (3x+2)\left[\sigma_1(3x+2)^{\frac{\sqrt{35}}{6}} + \sigma_2(3x+2)^{-\frac{\sqrt{35}}{6}}\right], \\
I_{pi;\,x} &= -\frac{4(3x+2)}{71}\sin[\log(3x+2)] \quad . \quad (5)
\end{aligned}
$$

12.18 Problems Group I, 7-chapt

[Note: Look just below (7.13).]

$$
\begin{aligned}
(1): \quad & y = \frac{x^3}{12} - x + \sigma_0\left[x^2 + \sigma_0\, x\right] + \sigma_1 \\
or \quad , \quad & y = \frac{x^3}{12} - x + \sigma_0\left[-x^2 + \sigma_0\, x\right] + \sigma_1 .
\end{aligned}
$$

12.19 Problems Group II, 7-chapt

[Note: Look at (7.28).]

$$
\begin{aligned}
(1): \quad & y = 2\,\sigma_0\,\tanh\left[\sigma_0\, x + \sigma_1\right] \\
(2): \quad & y = -\log\left[\sigma_0 - \sigma_1\,\exp\left(-\frac{x}{\sigma_0}\right)\right] .
\end{aligned}
$$

12.20 Problems Group III, 7-chapt

[Note: Look just below (7.47).]

$$
\begin{aligned}
(1): \quad & u_2(x) = \sigma_2\,\cos(x) \\
(2): \quad & u_2(x) = \sigma_2\,\exp\left[(-\sqrt{2}-1)\,x\right] \\
(3): \quad & u_2(x) = \sigma_2\,\exp\left(-\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) \\
(4): \quad & u_2(x) = \sigma_2\,\exp(x)\cos(x) \\
(5): \quad & u_2(x) = \sigma_2\, x^{(1-\sqrt{2})} \\
(6): \quad & u_2(x) = \sigma_2\,\exp\left(\frac{x^2}{2}\right)\cdot\int\exp\left(\frac{-x^2}{2}\right)\mathrm{d}x
\end{aligned}
$$

$$(7): \quad u_2(x) = \sigma_2 \exp\left(-x - \frac{x^2}{2}\right) \cdot \int \exp\left(\frac{x^2 + 4x}{2}\right) .$$

12.21 Problems Group IV, 7-chapt

[Note: Look at (7.62).]

$$(1): \quad I_{pi}(x) = \frac{\exp(x)}{2}$$
$$(2): \quad I_{pi}(x) = -\left(\frac{x^2}{2} + x + \frac{3}{2}\right)$$
$$(3): \quad I_{pi}(x) = \left(\frac{2}{3}\right)\left[x - \frac{2}{3}\right]$$
$$(4): \quad I_{pi}(x) = \log(x) + 1$$
$$(5): \quad I_{pi}(x) = \left(\frac{1}{2x}\right)$$
$$(6): \quad I_{pi}(x) = \left(\frac{x}{2}\right)\left[\log(x) - 1\right]$$
$$(7): \quad I_{pi}(x) = \left(\frac{1}{8x}\right)\left[2\log(x) + 1\right]$$
$$(8): \quad I_{pi}(x) = (x+1)\log(x+1) - (x+1)\log(x) - 1$$
$$(9): \quad I_{pi}(x) = \frac{1}{2}$$
$$(10): \quad I_{pi}(x) = \left(\frac{x}{2}\right) + \frac{1}{2} .$$

12.22 Problems Group V, 7-chapt

[Note: Look just below (7.100).]

Using the procedure outlined in (7.64)–(7.71), solution of differential equation obeyed by y is given below: as is the solution u.

$$(1): \quad y(x) = \sigma_1 \sin(x\sqrt{5}) + \sigma_2 \cos(x\sqrt{5}) + \frac{1}{5} \; ; \; u(x) = \exp(x^2)\, y(x).$$

$$(2): \quad y(x) = \sigma_1 \sin(x) + \sigma_2 \cos(x) + 1 \; ; \; u(x) = \exp\left(\frac{x^2}{2}\right) y(x).$$

$$(3): \quad y(x) = \sigma_1 \sin\left[\left(\frac{x}{2}\right)\sqrt{6}\right] + \sigma_2 \cos\left[\left(\frac{x}{2}\right)\sqrt{6}\right] + \frac{2}{3} \; ;$$

$$u(x) \;=\; \exp\left(\frac{3x^2}{4}\right)\, y(x).$$

$$(4)\;:\quad y(x) \;=\; \sigma_1\,(x)^{\frac{1}{2}+\frac{\sqrt{5}}{2}} \;+\; \sigma_2\,(x)^{\frac{1}{2}-\frac{\sqrt{5}}{2}} \;+\; x^{-1}\;;\;\; u(x) \;=\; x\,y(x).$$

12.23 Problems Group I, 11-chapt

[Note: Look at (11.31).]

All we have to do to solve (11.31) is to use in (11.24) their given values of α, β, γ. [Note: These symbols were defined in (11.5).] First few terms of the relevant Frobenius power series solution of (11.31)-(1)–(10) are the following.

For (1), $\alpha = 1$, $\beta = -1$, $\gamma = 1$. Therefore

$$(1)\;:\; u(x) \;=\; \sigma_1\left[1-\left(\frac{1}{2}\right)x^2+\left(\frac{5}{24}\right)x^4-\left(\frac{37}{720}\right)x^6+O(x^8)\right] + \;\sigma_2\left[x-\left(\frac{1}{3}\right)x^3+\left(\frac{7}{60}\right)x^5-\left(\frac{31}{1260}\right)x^7+O(x^9)\right]\;.$$

For (2), $\alpha = 1$, $\beta = -3$, $\gamma = 2$. Therefore

$$(2)\;:\; u(x) = \sigma_1\left[1-x^2+\left(\frac{7}{12}\right)x^4-\left(\frac{13}{60}\right)x^6+O(x^8)\right] +\sigma_2\left[x-\left(\frac{1}{2}\right)x^3+\left(\frac{11}{40}\right)x^5-\left(\frac{137}{1680}\right)x^7+O(x^9)\right]\;.$$

For (3), $\alpha = 1$, $\beta = 3$, $\gamma = -2$. Therefore

$$(3)\;:\; u(x) \;=\; \sigma_1\left[1+x^2-\left(\frac{1}{4}\right)x^4-\left(\frac{1}{12}\right)x^6+O(x^8)\right] + \;\sigma_2\left[x+\left(\frac{1}{6}\right)x^3-\left(\frac{19}{120}\right)x^5-\left(\frac{1}{1680}\right)x^7+O(x^9)\right]\;.$$

For (4), $\alpha = 4$, $\beta = -1$, $\gamma = -4$. Therefore

$$(4)\;:\; u(x) \;=\; \sigma_1\left[1+2\,x^2-\left(\frac{7}{12}\right)x^4+\left(\frac{3}{10}\right)x^6+O(x^8)\right] + \;\sigma_2\left[x+\left(\frac{1}{20}\right)x^5-\left(\frac{2}{105}\right)x^7+O(x^9)\right]\;.$$

For (5), $\alpha = -1$, $\beta = -1$, $\gamma = 1$. Therefore

$$(5) \ : \ u(x) \ = \ \sigma_1 \left[1 - \left(\frac{1}{2} \right) x^2 + \left(\frac{1}{24} \right) x^4 - \left(\frac{1}{80} \right) x^6 + O(x^8) \right] + \ \sigma_2 \left[x + \left(\frac{1}{20} \right) x^5 + \left(\frac{1}{210} \right) x^7 + O(x^9) \right] \ .$$

For (6), $\alpha = 1, \beta = 1, \gamma = 1$. Therefore

$$(6) \ : \ u(x) \ = \ \sigma_1 \left[1 + \left(\frac{1}{2} \right) x^2 + \left(\frac{5}{24} \right) x^4 + \left(\frac{37}{720} \right) x^6 + O(x^8) \right] + \ \sigma_2 \left[x + \left(\frac{1}{3} \right) x^3 + \left(\frac{7}{60} \right) x^5 + \left(\frac{1}{180} \right) x^7 + O(x^9) \right] \ .$$

For (7), $\alpha = -1, \beta = -1, \gamma = -1$. Therefore

$$(7) \ : \ u(x) \ = \ \sigma_1 \left[1 + \left(\frac{1}{2} \right) x^2 + \left(\frac{5}{24} \right) x^4 + \left(\frac{37}{720} \right) x^6 + O(x^8) \right] + \ \sigma_2 \left[x + \left(\frac{1}{3} \right) x^3 + \left(\frac{7}{60} \right) x^5 + \left(\frac{31}{1260} \right) x^7 + O(x^9) \right] \ .$$

For (8), $\alpha = 2, \beta = 2, \gamma = 2$. Therefore

$$(8) \ : \ u(x) \ = \ \sigma_1 \left[1 - x^2 + \left(\frac{1}{3} \right) x^4 - \left(\frac{2}{45} \right) x^6 + O(x^8) \right] + \ \sigma_2 \left[x - \left(\frac{2}{3} \right) x^3 + \left(\frac{1}{6} \right) x^5 - \left(\frac{1}{63} \right) x^7 + O(x^9) \right] \ .$$

For (9), $\alpha = -2, \beta = -2, \gamma = -2$. Therefore

$$(9) \ : \ u(x) \ = \ \sigma_1 \left[1 + x^2 + \left(\frac{2}{3} \right) x^4 + \left(\frac{13}{45} \right) x^6 + O(x^8) \right] + \ \sigma_2 \left[x + \left(\frac{2}{3} \right) x^3 + \left(\frac{11}{30} \right) x^5 + \left(\frac{43}{315} \right) x^7 + O(x^9) \right] \ .$$

For (10), $\alpha = 3, \beta = 3, \gamma = -3$. Therefore

$$(10) \ : \ u(x) \ = \ \sigma_1 \left[1 + \left(\frac{3}{2} \right) x^2 - \left(\frac{5}{8} \right) x^4 + \left(\frac{3}{80} \right) x^6 + O(x^8) \right] + \ \sigma_2 \left[x - \left(\frac{3}{20} \right) x^5 + \left(\frac{3}{70} \right) x^7 + O(x^9) \right] \ .$$

12.24 Problems Group II, 11-chapt

[Note: Look at (11.60).]

Information given in (11.52)–(11.58) is used to solve the five differential equations in problem set (II). These problems are similar in form to (b)-(11.32).

For (1), $\alpha = 1, \beta = 2, \gamma = 1, \mu = 3, \nu = 3, \rho = 1$. Therefore

$$\begin{aligned}(1)\ :\ u(x) &= \sigma_1\left[1 - \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{24} + \left(\frac{7}{30}\right)x^5\right] \\ &+ \sigma_1\left[\left(\frac{47}{720}\right)x^6 - \left(\frac{135}{2520}\right)x^7\right] + \sigma_2\left[x - \frac{x^2}{2} - \frac{x^3}{3} - \left(\frac{x^4}{24}\right)\right] \\ &+ \sigma_2\left[\frac{x^5}{10} + \left(\frac{81}{720}\right)x^6 - \left(\frac{29}{2520}\right)x^7\right] + O(x^8)\ .\end{aligned}$$

For (2), $\alpha = 0; \beta = 1; \gamma = 0; \mu = 2; \nu = 2; \rho = 0$. Therefore

$$\begin{aligned}(2)\ :u(x) &= \sigma_1\left[1 - \frac{x^3}{3} - \frac{x^4}{6} + \frac{x^5}{20} + \left(\frac{2}{45}\right)x^6 + \left(\frac{18}{1008}\right)x^7\right] \\ &+ \sigma_2\left[x - \frac{x^3}{6} - \frac{x^4}{6} + - \left(\frac{3}{40}\right)x^5 + \frac{x^6}{30} + \left(\frac{25}{1008}\right)x^7\right] + O(x^8)\ .\end{aligned}$$

For (3), $\alpha = 1; \beta = 0; \gamma = 1; \mu = 2; \nu = 1; \rho = 1$. Therefore

$$\begin{aligned}(3)\ :\ u(x) &= \sigma_1\left[1 - \frac{x^2}{2} - \frac{x^4}{8} + \frac{x^5}{10} + \left(\frac{5}{240}\right)x^6 + \frac{x^7}{105}\right] \\ &+ \sigma_2\left[x - \frac{x^2}{2} - \frac{x^4}{8} + \left(\frac{9}{240}\right)x^6 + \frac{x^7}{105}\right] + O(x^8)\ .\end{aligned}$$

For (4), $\alpha = 0; \beta = 2; \gamma = 2; \mu = 0; \nu = 0; \rho = 1$. Therefore

$$\begin{aligned}(4)\ :\ u(x) &= \sigma_1\left[1 - \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{24} - \left(\frac{2}{15}\right)x^5 + \left(\frac{13}{504}\right)x^6 + \left(\frac{23}{720}\right)x^7\right] \\ &+ \sigma_2\left[x - x^2 + \frac{x^3}{6} + \frac{x^4}{3} - \left(\frac{23}{120}\right)x^5 + \left(\frac{61}{1008}\right)x^6 - \left(\frac{13}{360}\right)x^7\right] \\ &+ O(x^8)\ .\end{aligned}$$

For (5), $\alpha = 2; \beta = 1; \gamma = 1; \mu = 5; \nu = 1; \rho = 4$. Therefore

$$\begin{aligned}(5)\ :\ u(x) &= \sigma_1\left[1 - 2x^2 + \frac{x^3}{2} + \left(\frac{11}{24}\right)x^4 + \left(\frac{7}{30}\right)x^5 + \frac{x^6}{18} - \left(\frac{1087}{5040}\right)x^7\right] \\ &+ \sigma_2\left[x - \frac{x^2}{2} - \left(\frac{2}{3}\right)x^3 + \frac{x^4}{6}\right]\end{aligned}$$

$$+\sigma_2\left[\left(\frac{3}{40}\right)x^5+\left(\frac{131}{720}\right)x^6+\left(\frac{8}{5040}\right)x^7\right]+O(x^8)\ .$$

12.25 Problems Group III, 11-chapt

[Note: Look at (11.97).]

We use (11.85), set $n=2,3,4,5,6,7$ in (11.86), and solve the following differential equations that are of the form (11.78). Only the indices β,γ,ν,ρ were given in (11.97)-(1)–(5). For convenience, they are reprinted below.

$$\begin{aligned}
&(1)\ :\ [\beta=-3\,;\ \gamma=3\,;\ \nu=2\,;\ \rho=-4\,]\ .\\
&(2)\ :\ [\beta=2\,;\ \gamma=-3\,;\ \nu=1\,;\ \rho=-2\,]\ .\\
&(3)\ :\ [\beta=3\,;\ \gamma=-2\,;\ \nu=-2\,;\ \rho=-3\,]\ .\\
&(4)\ :\ [\beta=1\,;\ \gamma=-2\,;\ \nu=-1\,;\ \rho=-1\,]\ .\\
&(5)\ :\ [\beta=-1\,;\ \gamma=-5\,;\ \nu=-1\,;\ \rho=-4\,]\ .
\end{aligned}$$

For (1), $\beta=-3\,;\gamma=3\,;\nu=2\,;\rho=-4$ and the solutions are

$$\begin{aligned}
(1a)\ &:\left[\frac{u(x)}{\sigma_1 x^{-1+\sqrt{5}}}\right]\\
&=\ 1+\frac{x}{19}(35-13\sqrt{5})+\frac{x^2}{76}(230-99\sqrt{5})+\frac{x^3}{2,508}(9,675-4,283\sqrt{5})\\
&+\ \frac{x^4}{20,064}(110,555-49,331\sqrt{5})+\frac{x^5}{25,080}(15,306-6,805\sqrt{5})\\
&+\ \frac{x^6}{109,440}(23,965-10,671\sqrt{5})+\frac{x^7}{5,554,080}(432,160-192,689\sqrt{5})+\cdots
\end{aligned}$$

$$\begin{aligned}
(1b)\ &:\left[\frac{u(x)}{\sigma_1 x^{-1-\sqrt{5}}}\right]\\
&=\ 1+\frac{x}{19}(35+13\sqrt{5})+\frac{x^2}{76}(230+99\sqrt{5})+\frac{x^3}{2,508}(9,675+4,283\sqrt{5})\\
&+\ \frac{x^4}{20,064}(110,555+49,331\sqrt{5})+\frac{x^5}{25,080}(15,306+6,805\sqrt{5})\\
&+\ \frac{x^6}{109,440}(23,965+10,671\sqrt{5})+\frac{x^7}{5,554,080}(432,160+192,689\sqrt{5})+\cdots
\end{aligned}$$

For (2), $\beta=2\,;\gamma=-3\,;\nu=1\,;\rho=-2$ and the solutions are

$$
\begin{aligned}
(2a) \ : & \left[\frac{u(x)}{\sigma_1 x^{2+\sqrt{6}}}\right] \\
= \ & 1-\frac{x}{23}(19+8\sqrt{6})+\frac{x^2}{92}(67+27\sqrt{6})-\frac{x^3}{1,380}(581+241\sqrt{6}) \\
+ \ & \frac{x^4}{22,080}(4,312+1,657\sqrt{6})+\frac{x^5}{22,080}(-3,544+211\sqrt{6}) \\
+ \ & \frac{x^6}{264,960}(42,782-12,953\sqrt{6})+\frac{x^7}{1,854,720}(-224,746+83,447\sqrt{6})+\cdots
\end{aligned}
$$

$$
\begin{aligned}
(2b) \ : & \left[\frac{u(x)}{\sigma_1 x^{2-\sqrt{6}}}\right] \\
= \ & 1-\frac{x}{23}(-19+8\sqrt{6})+\frac{x^2}{92}(67-27\sqrt{6})+\frac{x^3}{1,380}(-581+241\sqrt{6}) \\
+ \ & \frac{x^4}{22,080}(4,312-1,657\sqrt{6})-\frac{x^5}{22,080}(3,544+211\sqrt{6}) \\
+ \ & \frac{x^6}{264,960}(42,782+12,953\sqrt{6})-\frac{x^7}{1,854,720}(224,746+83,447\sqrt{6})+\cdots
\end{aligned}
$$

For (3), $\beta = 3\,;\, \gamma = -2\,;\, \nu = -2\,;\, \rho = -3$ and the solutions are

$$
\begin{aligned}
(3a) \ : & \left[\frac{u(x)}{\sigma_1 x^{\frac{3+\sqrt{21}}{2}}}\right] \\
= \ & 1-\frac{x}{20}(29+\sqrt{21})+\frac{x^2}{680}(647+93\sqrt{21})-\frac{x^3}{12,240}(5,847+1,573\sqrt{21}) \\
+ \ & \frac{x^4}{288}(114+11\sqrt{21})+\frac{x^5}{5,760}(-3,057+347\sqrt{21}) \\
+ \ & \frac{x^6}{172,800}(95,543-17,163\sqrt{21})+\frac{x^7}{2,419,200}(-1,021,789+205,549\sqrt{21})+\cdots
\end{aligned}
$$

$$
\begin{aligned}
(3b) \ : & \left[\frac{u(x)}{\sigma_1 x^{\frac{3-\sqrt{21}}{2}}}\right] \\
= \ & 1+\frac{x}{20}(-29+\sqrt{21})+\frac{x^2}{680}(647-93\sqrt{21})+\frac{x^3}{12,240}(-5,847+1,573\sqrt{21}) \\
+ \ & \frac{x^4}{288}(114-11\sqrt{21})-\frac{x^5}{5,760}(3,057+347\sqrt{21}) \\
+ \ & \frac{x^6}{172,800}(95,543+17,163\sqrt{21})-\frac{x^7}{2,419,200}(1,021,789+205,549\sqrt{21})+\cdots
\end{aligned}
$$

For (4), $\beta = 1\,;\, \gamma = -2\,;\, \nu = -1\,;\, \rho = -1$ and the solutions are

$$(4a)\ :\left[\frac{u(x)}{\sigma_1 x^{\frac{3+\sqrt{13}}{2}}}\right]$$
$$= 1-\frac{x}{2}+\frac{x^2}{72}(7+\sqrt{13})-\frac{x^3}{144}(1+\sqrt{13})$$
$$+ \frac{x^4}{288}(-2+\sqrt{13})+\frac{x^5}{8,640}(29-10\sqrt{13})$$
$$+ \frac{x^6}{88,320}(-81+25\sqrt{13})+\frac{x^7}{3,709,440}(689-203\sqrt{13})+\cdots$$

$$(4b)\ :\left[\frac{u(x)}{\sigma_2 x^{\frac{3-\sqrt{13}}{2}}}\right]$$
$$= 1-\frac{x}{2}-\frac{x^2}{72}(-7+\sqrt{13})+\frac{x^3}{144}(-1+\sqrt{13})$$
$$- \frac{x^4}{288}(2+\sqrt{13})+\frac{x^5}{8,640}(29+10\sqrt{13})$$
$$- \frac{x^6}{88,320}(81+25\sqrt{13})+\frac{x^7}{3,709,440}(689+203\sqrt{13})+\cdots$$

For (5), $\beta=-1$; $\gamma=-5$; $\nu=-1$; $\rho=-4$ and the solutions are

$$(5a)\ :\left[\frac{u(x)}{\sigma_1 x^{3+\sqrt{13}}}\right]$$
$$= 1+\frac{x}{51}(22+7\sqrt{13})+\frac{x^2}{68}(15+4\sqrt{13})+\frac{x^3}{8,772}(588+167\sqrt{13})$$
$$+ \frac{x^4}{631,584}(10,267+2,773\sqrt{13})+\frac{x^5}{1,052,640}(3,121+915\sqrt{13})$$
$$+ \frac{x^6}{743,040}(407+87\sqrt{13})+\frac{x^7}{15,603,840}(-3,205+1,463\sqrt{13})+\cdots$$

$$(5b)\ :\left[\frac{u(x)}{\sigma_1 x^{3-\sqrt{13}}}\right]$$
$$= 1+\frac{x}{51}(22-7\sqrt{13})+\frac{x^2}{68}(15-4\sqrt{13})+\frac{x^3}{8,772}(588-167\sqrt{13})$$
$$+ \frac{x^4}{631,584}(10,267-2,773\sqrt{13})+\frac{x^5}{1,052,640}(3,121-915\sqrt{13})$$
$$+ \frac{x^6}{743,040}(407-87\sqrt{13})-\frac{x^7}{15,603,840}(3,205+1,463\sqrt{13})+\cdots$$

Chapter 13
Answer to Additional Assigned Problems



13.1 Fourier Transforms

A function $f(t)$ that is not necessarily periodic but is reasonably well behaved can be represented in terms of an integral involving its Fourier transform.

$$f(t) = \left(\frac{1}{2\pi}\right) \int_{-\infty}^{\infty} F(\omega) \exp(i\,\omega\, t) d\omega \quad , \tag{13.1}$$

where

$$F(\omega) = \int_{-\infty}^{\infty} f(t) \exp(-i\,\omega\, t) \mathrm{d}t \quad . \tag{13.2}$$

Often $F(\omega)$ and $f(t)$ are referred to as the inverse Fourier transform of each other.

13.2 Examples Set (I) and Solution

Work out Fourier transform of six functions given below.

$$\begin{aligned} c_1\, f_1(t) + c_2\, f_2(t)\,. \quad & (1) \\ f^*(t)\,. \quad & (2) \\ f'(t)\,. \quad & (3) \\ For\ \alpha > 0\,,\ f(\alpha\, t)\,. \quad & (4) \\ f(t - t_0)\,. \quad & (5) \\ \exp(i\omega_0 t)\, f(t)\,. \quad & (6) \end{aligned} \tag{13.3}$$

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13.3 Solution to Examples Set (I) Fourier transforms

(1) : Consider $c_1\, f_1(t) + c_2\, f_2(t) = (1/(2\pi)\, \int_{-\infty}^{\infty}[c_1\, F_1(\omega) + c_2\, F_2(\omega)] \exp(i\, \omega\, t)\, d\omega$. The right-hand side of the above equation leads to the conclusion that the relevant Fourier transform for $c_1\, f_1(t) + c_2\, f_2(t)$ is
$c_1\, F_1(\omega) + c_2\, F_2(\omega)$. (1)

(2) : Consider $f(t) = 1/(2\pi)\, \int_{-\infty}^{\infty} F(\omega) \exp(i\, \omega\, t)\, d\omega$.
Take complex conjugate of both sides of the above equation.
$f^*(t) = 1/(2\pi)\, \int_{-\infty}^{\infty} F^*(\omega) \exp(-i\, \omega\, t)\, d\omega$. Set $\omega = -\omega_0$.
$= -1/(2\pi)\, \int_{\infty}^{-\infty} F^*(-\omega_0) \exp(i\, \omega_0\, t)\, d\omega_0$. Now set $\omega_0 = \omega$. We get
$f^*(t) = 1/(2\pi)\, \int_{-\infty}^{\infty} F^*(-\omega) \exp(i\, \omega\, t)\, d\omega$.
Therefore, the relevant Fourier transform for $f^*(t)$ is $F^*(-\omega)$. (2)

(3) : Consider $f(t) = 1/(2\pi)\, \int_{-\infty}^{\infty} F(\omega) \exp(i\, \omega\, t)\, d\omega$.
Differentiate both sides with respect to t.
$f'(t) = 1/(2\pi)\, \int_{-\infty}^{\infty} (i\omega) F(\omega) \exp(i\, \omega\, t)\, d\omega$.
Therefore, the relevant Fourier transform for $f'(t)$ is $i\, \omega F(\omega)$. (3)

(4) : Consider $f(t') = 1/(2\pi)\, \int_{-\infty}^{\infty} F(\omega') \exp(i\, \omega'\, t')\, d\omega'$.
Use $t' = \alpha\, t$ and write $f(\alpha\, t) = 1/(2\pi)\, \int_{-\infty}^{\infty} F(\omega') \exp(i\, \omega'\, \alpha\, t)\, d\omega'$.
Setting $\omega'\, \alpha = \omega$ gives $f(\alpha\, t) = 1/(2\pi)\, \int_{-\infty}^{\infty} F(\omega/\alpha) \exp(i\, \omega'\, \alpha\, t)\, d\omega/\alpha$.
Therefore, the relevant Fourier transform for $f(\alpha\, t)$ where $\alpha > 0$ is $F(\omega/\alpha)/\alpha$. (4)

(5) : Consider $f(t') = 1/(2\pi)\, \int_{-\infty}^{\infty} F(\omega) \exp(i\, \omega\, t')\, d\omega$.
Use $t' = t - t_0$ and write $f(t - t_0) = 1/(2\pi)\, \int_{-\infty}^{\infty} F(\omega) \exp[i\, \omega\, (t - t_0)]\, d\omega$.
Rewriting gives: $f(t - t_0) = 1/(2\pi)\, \int_{-\infty}^{\infty} [F(\omega) \exp(-i\omega\, t_0)] \exp(i\, \omega\, t\, d\omega$.
Inverting the above—that is,
$[F(\omega) \exp(-i\, \omega\, t_0)] \;=\; \int_{-\infty}^{\infty} f(t - t_0) \exp(-i\, \omega\, t) \mathrm{d}t$—
leads to the result that the Fourier transform for $f(t - t_0)$ is $F(\omega) \exp(-i\omega\, t_0)$. (5)

(6) : In order to determine the Fourier transform, $F_0(\omega)$, of the function $[f(t) \exp(i\, \omega_0\, t)]$ proceed as follows: Write:

$$\begin{aligned} F_0(\omega) &= 1/(2\pi)\, \int_{-\infty}^{\infty} [f(t) \exp(i\, \omega_0\, t)] \exp(-i\, \omega\, t)\, \mathrm{d}t \\ &= 1/(2\pi)\, \int_{-\infty}^{\infty} f(t) \exp(-i\, [\omega - \omega_0]\, t)\, \mathrm{d}t \\ &= F(\omega - \omega_0)\, . \end{aligned} \tag{13.4}$$

Clearly, therefore, the Fourier transform of $[f(t) \exp(i\, \omega_0\, t)]$ is $F(\omega - \omega_0)$. (6)

13.4 Dirac's Delta Function

It is necessary to give some description of the function $\delta(x-a)$. Indeed, Dirac's delta function deserves a detailed and thorough review—see for instance, Dean G. *Duffy*[24]. Still, the following relationships should suffice for the current needs.

For real a

$$\begin{aligned}\delta(t-a) &= \infty\ ,\ when\ x=a\\ &= 0\ ,\ when\ x\neq a\ ,\end{aligned} \tag{13.5}$$

and

$$\int_{-\infty}^{\infty}\delta(t-a)\,\mathrm{d}t = 1\quad , \tag{13.6}$$

$$\int_{-\infty}^{\infty}\delta(t-a)f(t)\mathrm{d}t = f(a)\ . \tag{13.7}$$

A choice popular with electrical engineers is to relate the delta function to the derivative of Heaviside step function $H(t-a)$. The Heaviside step function is defined for $a\geq 0$ as

$$\begin{aligned}H(t-a) &= 1,\quad for\ \ t>a\\ &= 0,\quad for\ \ t<a\ .\end{aligned} \tag{13.8}$$

And its relationship to the delta function is as follows.

$$\delta(t) = \frac{dH(t)}{\mathrm{d}t}\ . \tag{13.9}$$

Upon integration, (13.9) yields

$$H(t) = \int_{-\infty}^{t}\delta(x)\,\mathrm{d}x\ . \tag{13.10}$$

Another useful choice for $\delta(x)$ is one recommended by T. B. *Boykin*[35]. That is

$$\delta(x) = \frac{1}{2L} + \frac{1}{L}\sum_{n=1}^{\infty}\cos\left(\frac{n\pi x}{L}\right)\quad . \tag{13.11}$$

In order to check whether the Boykin relationship (13.11) is valid, we examine the accuracy of its following prediction.

$$\int_{-L}^{L} f(x)\,\delta(x)\,\mathrm{d}x \;=\; f(0)\;. \tag{13.12}$$

That is, we check the following relationship for validity:

$$\begin{aligned}
&\int_{-L}^{L} f(x)\left[\frac{1}{2\,L}+\frac{1}{L}\sum_{n=1}^{\infty}\cos\left(\frac{n\pi\,x}{L}\right)\right]\mathrm{d}x\\
&=\int_{-L}^{L} f(x)\left[\frac{1}{2\,L}\right]\mathrm{d}x+\int_{-L}^{L} f(x)\left[\frac{1}{L}\sum_{n=1}^{\infty}\cos\left(\frac{n\pi\,x}{L}\right)\right]\mathrm{d}x\\
&=f(0)\;.
\end{aligned} \tag{13.13}$$

In other words, we examine whether (13.13) indeed holds true—see (13.15) for the appropriate result for $f(0)$. That is

$$f(0)=a_0+\sum_{n=1}^{\infty}a_n\;. \tag{13.14}$$

However, before that can be done the function $f(x)$ needs to be represented in a suitable format.

To that end, proceed as follows. Given the function $f(x)$ is piecewise differentiable with period 2 L in the range [-L,L], it can be represented by an infinite Fourier series.

$$f(x)=a_0+\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\,\pi\,x}{L}\right)+b_n\,\sin\left(\frac{n\,\pi\,x}{L}\right)\right]\;. \tag{13.15}$$

The $f(x)$ given above in (13.15) is now inserted into (13.13). Upon working out such (13.13), we find that the result is the same as that predicted by the use of Boykin's equation: Meaning, it is equal to $f(0)$. This fact can be confirmed by comparison with (13.16).

$$\begin{aligned}
&\int_{-L}^{L}\left\{a_0+\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\,\pi\,x}{L}\right)+b_n\,\sin\left(\frac{n\,\pi\,x}{L}\right)\right]\right\}\left[\frac{1}{2\,L}\right]\mathrm{d}x\\
&+\int_{-L}^{L}\left\{a_0+\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\,\pi\,x}{L}\right)+b_n\,\sin\left(\frac{n\,\pi\,x}{L}\right)\right]\right\}\left[\frac{1}{L}\sum_{n=1}^{\infty}\cos\left(\frac{n\pi\,x}{L}\right)\right]\mathrm{d}x\\
&=a_0+\sum_{n=1}^{\infty}a_n\;=\;f(0)\;.\quad\text{Q.E.D}
\end{aligned} \tag{13.16}$$

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